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ANALYSIS OF STUDENT ACHIEVEMENTS IN TEACHING COMPLEX NUMBERS USING GEOGEBRA SOFTWARE

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Abstract. The research deals with the analysis of students' achievements at the level of complex numbers in a "traditional" way and teaching enriched with multimedia software. The paper focuses on teaching complex numbers using the GeoGebra/HotPotatoes multimedia software. This approach to teaching aims to increase student activity and engagement, but also to raise the teaching process to a higher level of achievements and motivate students to work and learn more independently.

1. Introduction

Today, the technology is used more and more, it is necessary to adapt the teaching process accordingly. An important role has to be given to linkage between images and certain conditions, in order for the students to develop their knowledge [1], [2]. Multimedia approach in teaching of mathematics can be very useful in explanation of mathematical ideas, abstract terms and evaluation of knowledge.

In secondary schools of the Republika Srpska (Bosnia and Herzegovina), the curriculum for mathematics deals with the subject of complex numbers. Secondary school is the first time students are informed about complex numbers. As the name suggests, this topic is quite complex and demanding, since these are not the kind of numbers students face in their daily life. Through our extensive experience, we have come to a conclusion that students have a hard time adopting the lessons from this topic and, thus, lack motivation to further explore the application of complex numbers. In order to make the term more familiar and give a geometrical interpretation of a complex number, an idea was born to step back from the classical approach to this topic and use mathematical software called GeoGebra. This software is useful for visualisation of the term of a complex number, as well as the presentation of all operations over the field of complex numbers. Given that the software is simple to use, free and available in Serbian language, it makes it much more advantageous compared to other similar software. Therefore, GeoGebra software can be applied in different forms of mathematics teaching [3].

Key words. Complex number, Gauss plane, imaginary unit, geometric interpretation.

2. Teaching complex numbers in a “traditional” manner

In order to cover the topic it is necessary to have 5 lessons during which the students are informed about: the term of a complex number, set of complex numbers, and presentation of complex numbers in a Gauss plane, arithmetic operations and their application. The experience has shown that the emphasis in teaching complex numbers is often put on mechanical adoption of the procedures for solving problems from this topic. The biggest problem that occurs during the traditional way of treating this subject is the lack of deeper understanding of the geometrical interpretation of a complex number. This is the reason why such an approach has to be changed with a new one, which requires the use of GeoGebra.

3. Teaching complex numbers in Geogebra software

Teaching mathematics in GeoGebra is interesting because it assesses the level and method of application and the unique characteristics of teaching mathematics and computer science courses with the aim of improving the general context of learning and improving the digital competencies of students, that is, it uses the advantages of this teaching system compared to the traditional system [4]. GeoGebra is mathematical software that successfully links geometry, algebra, analysis and other areas. GeoGebra software is often used in the teaching of mathematics [5]-[9]. It is suitable for presentation and better understanding of mathematical content. GeoGebra has three ways of representation of mathematical objects: graphical, algebraic and table presentation. All three ways of representation of an object are dynamically linked and are automatically adjusted to each change that occurs in any of the representations, regardless of the way the object was created. The power of visualisation can be used as a mean for development of theoretical meaning of geometrical terms. GeoGebra can be installed on a computer, or can be used in an on-line mode [10]. Using GeoGebra software teaching and evaluation processes in addition to mathematical knowledge, they also include students' ICT (Information and Communications Technology) knowledge and skills [11].

3.1. The term of a complex number

During the first lesson, by solving an equation (1):

$$x^2 + a = 0 \tag{1}$$

The students are encouraged to conclude that given equations cannot always have a solution in the set of real numbers. This is the reason for the introduction of an *imaginary unit* and an algebraic form of a complex number (2).

$$z = a + bi, \qquad a, b \in \mathbb{R} \tag{2}$$

Each complex number has its geometrical interpretation. Just as all real numbers can be represented by an infinite straight line, in the same way the area of real and imaginary numbers can be represented by an infinite plane.

Number $z = a + bi$ can be represented in the Gauss plane as a point with coordinates *(a,b)* where the first coordinate of the ordered pair is the abscissa *a* (real part of the number) and the second coordinate of the ordered pair is the ordinate *b* (imaginary part of the number). It is possible to show this procedure in GeoGebra (Figure 1).

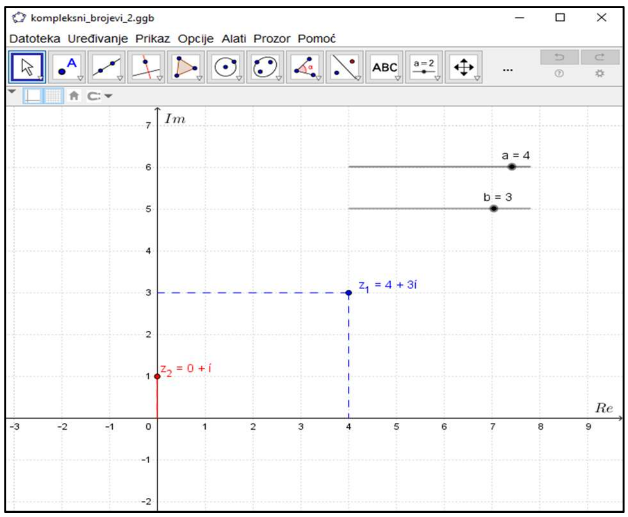


Figure 1. Display of complex number in the Gauss plane

By moving the sliders *a* and *b* the complex number in the Gauss plane is changed, so the students can see many examples in a short period of time and, if the class is held in a computer classroom, which is desirable in this case, they can explore for themselves.

3.2. Complex conjugate of a number

As a complex number is represented by a point in the Gauss plane, the distance between the observed point and the origin is defined as a modulus of the complex number and is calculated by the formula (3).

$$|z| = \sqrt{a^2 + b^2} \tag{3}$$

The following term is defined as a complex conjugate. If $z = a + bi$ is denoted as a complex number, then its complex conjugate is in formula (4):

$$\bar{z} = a - bi \tag{4}$$

Complex conjugate of a complex number is made when the sign of the imaginary part of the complex number is changed [12]. GeoGebra is suitable software for the representation of a complex number and it's conjugate.

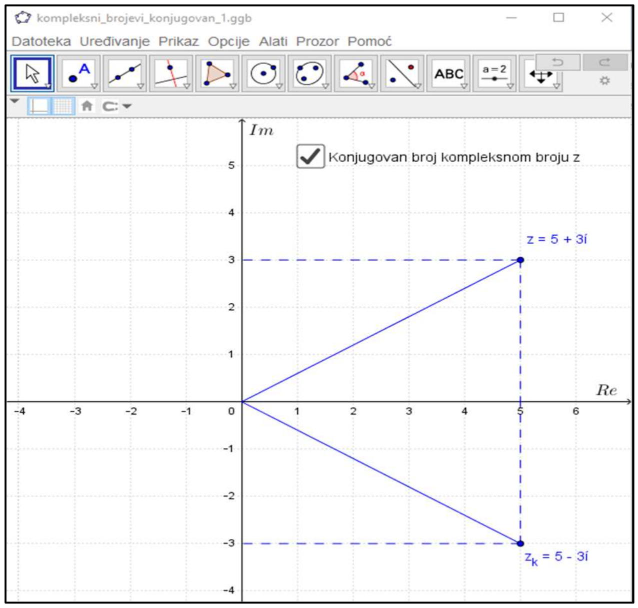


Figure 2. Display of a complex conjugate

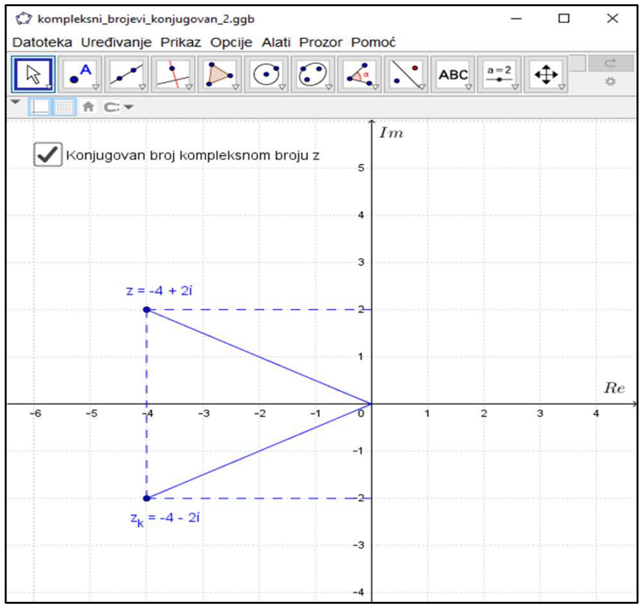


Figure 3. Display of complex conjugate

This is a simple way to show students *a conjugate of a complex number*. By showing students the geometric representation of a complex number and its conjugate, they can clearly see that their moduli are the same. This means that

$$|z| = |\bar{z}|$$

The advantage of GeoGebra is that, by moving point *z* we automatically get a new$\bar{z}$. This enables students to see many examples in a short time and notice the properties of a complex number and its conjugate (Figure 2. and Figure 3.)

3.3. Operations with complex numbers

After the introduction of algebraic and geometric form of a complex number, it is necessary to define binary operations over the field of complex numbers $\mathbb{C}$.

Let $z_1 = a + bi$ and $z_2 = c + di \quad a, b, c, d \in \mathbb{R}$
be complex numbers. Arithmetic operations are defined in the following way:

$$z_1 + z_2 = (a + c) + (b + d)i$$
$$z_1 - z_2 = (a - c) + (b - d)i$$
$$z_1 \cdot z_2 = (ac - bd) + (ad + bc)i$$
$$\frac{z_1}{z_2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i, \qquad z_2 \neq 0$$

The field of complex numbers is closed under all these operations, which means that the result of an operation between complex numbers is a complex number [13].

GeoGebra has also proven to be suitable for the processing and interpretation of the operations over the field of complex numbers. It is easy for the student to see in a Gauss plane that the sum, difference, product and quotient of complex numbers is a complex number (Figure 4.) It is important to stress that, besides being able to see the *geometric interpretation* of the newly formed complex number (including the modulus), by manipulating the points in the plane, it is possible to present these operations for different values of complex numbers. Also, it is important to mention that this software enables us to simplify complicated algebraic expressions consisted of several arithmetic operations and present their values geometrically.

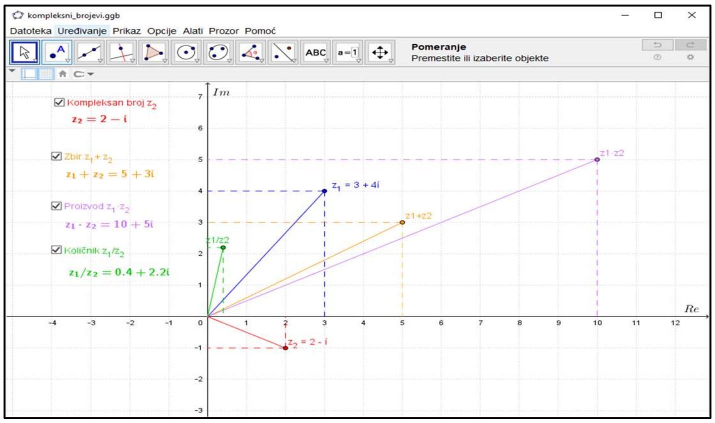


Figure 4. Operations with complex numbers

This feature helps the users find the solution easier and see its visualisation.

3.4. Power of the imaginary unit

Imaginary unit is the solution of the equation (5).

$$x^2 + 1 = 0 \tag{5}$$

It is denoted by i and its value is $i^2 = -1$. Geometric representation of the imaginary unit is point (0,1) [14]. Value of the imaginary unit can be checked in the following way:

$$i^2 = (0,1)(0,1) = (-1,0) = -1$$

It is obvious that:

$i^{4k} = 1$
$i^{4k+1} = i$
$i^{4k+2} = -1$
$i^{4k+3} = -i$, $k = 0,1,2,\dots$

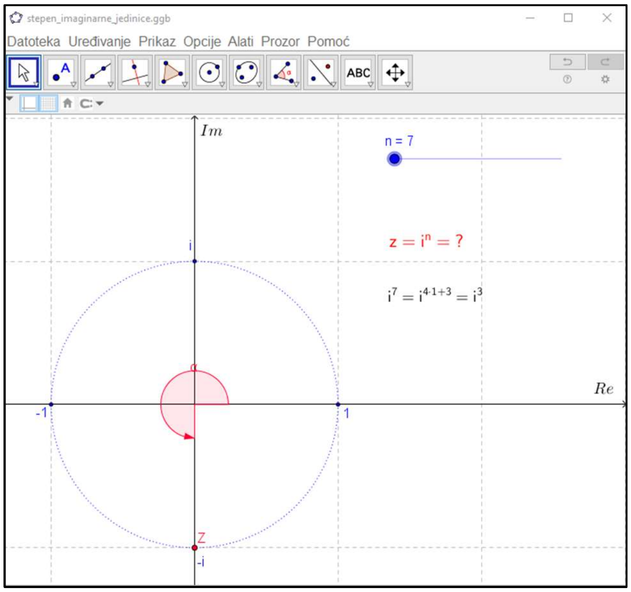


Figure 5. The n-th – power of number i

Figure 5. shows a graphical representation of the power of the imaginary unit. Movement of the slider easily represents the link between the power of the imaginary unit and the position of the corresponding point on the unit circle. By changing the exponent, the point is rotated in a positive direction by the angle of $\frac{\pi}{2}$, meaning that there are 4 different values of the power of the imaginary unit. This interpretation in GeoGebra enables the students to understand cyclical repetition of the imaginary unit power value.

3.5. Performance tasks

Example 1

The following complex numbers are given:
$z_1 = 3 - 4$ and $z_2 = 2 - i$.
Find the modulus of the complex number $\frac{z_1 \cdot z_2}{\bar{z}_1}$.

Solutions:

1. $z_1 \cdot z_2 = 2 - 11i$
2. $\bar{z}_1 = 3 + 4i$
3. $\frac{z_1 \cdot z_2}{\bar{z}_1} = -1{,}52 - 1{,}64i$
4. $\left|\frac{z_1 \cdot z_2}{\bar{z}_1}\right| = \sqrt{5} \approx 2{,}24$

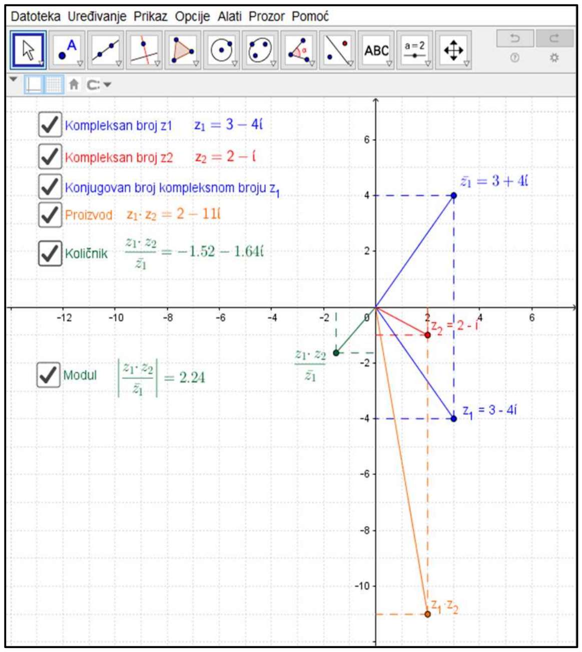


Figure 6. Graphical representation of Example 1

Figure 6 shows the steps in solving the given example. Therefore, the image shows complex numbers $z_1$andz_2. Their product, complex conjugate of z_1 quotient of the product and conjugate. Finally, the modulus of the complex number is clearly visible. Classical way of solving this task does not provide a possibility of visual representation of each step. GeoGebra saves time and enables us to check the solution itself.

Example 2

The following complex numbers are given:

$z_1 = 1 - i$ and $z_2 = -1 - i$. Find the value of expression$\left(\frac{z_1}{z_2}\right)^{2018}$.

Solution:

$$\frac{z_1}{z_2} = i$$

$$\left(\frac{z_1}{z_2}\right)^{2018} = i^{2018} = i^{4 \cdot 504 + 2} =$$
$$= (i^4)^{504} i^2 = 1 \cdot (-1) = -1$$

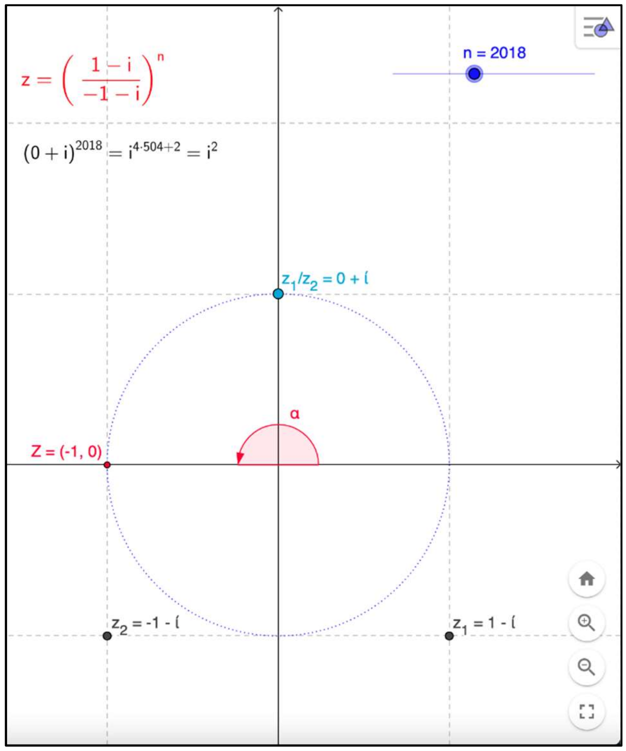


Figure 7. Graphical representation of Example 2

Figure 7. shows graphical solution of the example. Each solution step is shown with its geometric and algebraic interpretation, which helps the user understand the operations with complex numbers better. Besides the solution of this example, it is possible to use the slider and change the value of the exponent and get a number of similar examples.

3.6. GeoGebra CAS view

CAS (Computer Algebra System) view enables the users to work with algebraic expressions, functions, equations, matrices, numbers and datasets. In a simple way, this GeoGebra view enables solving equations, factoring of polynomials, differential and integral calculus. Since we have already seen that complex numbers can be represented in this software, all the above mentioned problems can be solved over the field of complex numbers with the use of CAS window.

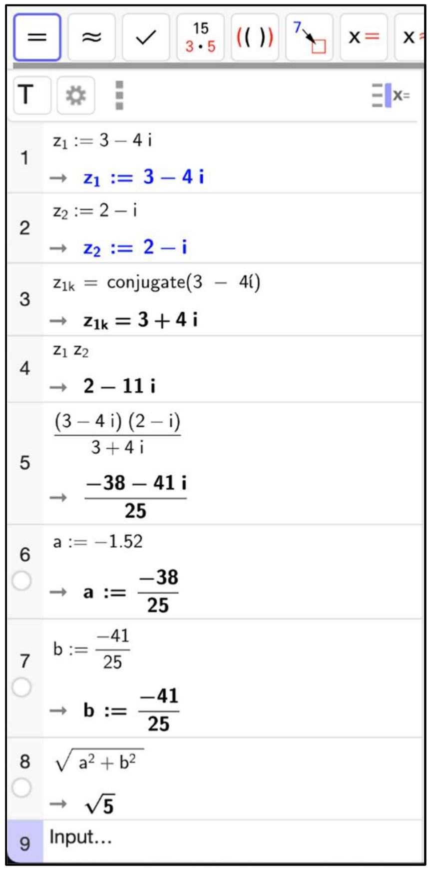


Figure 8. CAS view of Example 1

This GeoGebra option enables the users to find the solution of a problem in short time, or to check an existing solution. It is very important to highlight that the use of CAS view for the complex numbers requires for the imaginary unit to be defined.

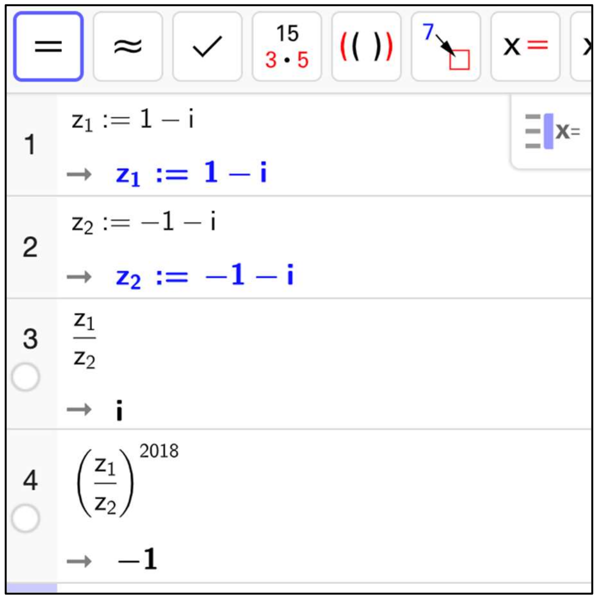


Figure 9. CAS view of Example 2

GeoGebra does not recognize i as the imaginary unit (Figure 8. and Figure 9.).

4. Results and discussion

The purpose of this research is to analyse the achievements of students who use the GeoGebra/HotPotatoes multimedia software. Teaching mathematics in this case is done by complex numbers. The research was conducted in the Secondary School of Economics in Doboj, Republika Srpska (Bosnia and Herzegovina), among the students of the second year, profile of economic technician, 220 students participated in the research.

The survey was conducted in the following chronological order:

1. Two classes were selected by the following criteria. The number of students in both classes was equal; the average grade in mathematics was approximately the same.
2. In Class A GeoGebra software was used during the elaboration of lesson of complex numbers. Class B processed the topic in the traditional way.
3. Upon the completion of the topic elaboration, both classes were tested. Students had 45 minutes to solve 10 performance tasks from the topic. The tasks included both, theoretical and arithmetic problems. Both classes did the test with the same problems, in the same period of time, on the same day (Figure 10a and Figure 10b.).
4. The test was presented electronically, using HotPotatoes software. Students solved the problems on a sheet of paper and then entered the final answers and solutions into the on-line test.

5. Upon entering their answers in HotPotatoes, the students had an opportunity to see whether the answer was correct, which gave them an opportunity to self-evaluate their test, prior to the evaluation made by their teacher.
6. Upon the evaluation of the test, the students were presented with automatically generated results of each task, as well as the final test score.

The advantage of this on-line test is that the students can compare their self-evaluation with the teacher's one.

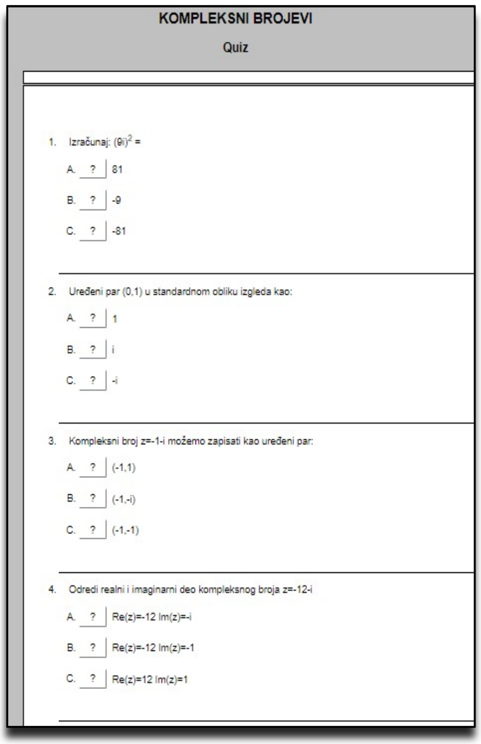
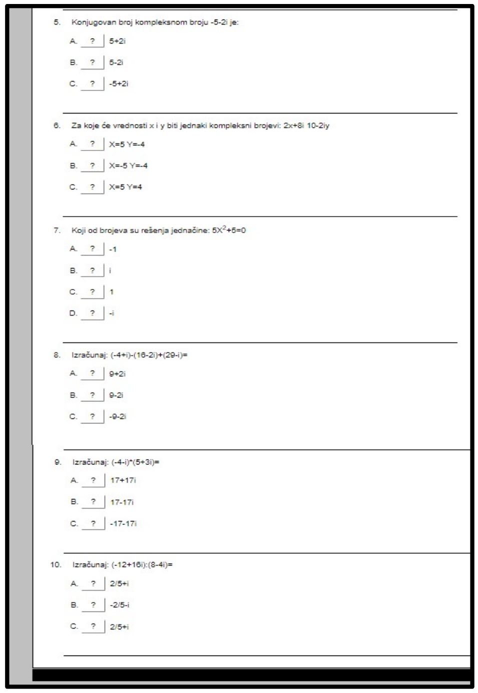


Figure 10a. Review of the first page of the test to check knowledge of complex numbers



5. Konjugovan broj kompleksnom broju -5-2i je:
A. ? 5+2i
B. ? 5-2i
C. ? -5+2i

6. Za koje će vrednosti x i y biti jednaki kompleksni brojevi: 2x+8i 10-2iy
A. ? X=5 Y=-4
B. ? X=-5 Y=-4
C. ? X=5 Y=4

7. Koji od brojeva su rešenja jednačine: $5X^2+5=0$
A. ? -1
B. ? i
C. ? 1
D. ? -i

8. Izračunaj: (-4+i)-(16-2i)+(29-i)=
A. ? 9+2i
B. ? 9-2i
C. ? -9-2i

9. Izračunaj: (-4-i)*(5+3i)=
A. ? 17+17i
B. ? 17-17i
C. ? -17-17i

10. Izračunaj: (-12+16i):(8-4i)=
A. ? 2/5+i
B. ? -2/5-i
C. ? 2/5+i

Figure 10b. Review of the second page of the test to check knowledge of complex numbers

A detailed analysis of the tests made by the class that used GeoGebra and the class that processed the topic in a traditional way produced the following results, shown in Figure 11.

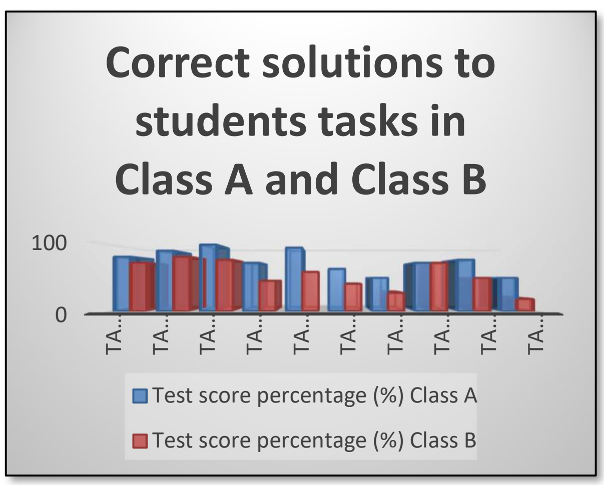


Figure 11. Test results for checking knowledge in the area of complex numbers

Out of possible 220 correct answers, Class A produced 166 (75.45%), while Class B produced 120 (54.54%) correct answers. We can conclude that Class A produced 20. 91% better results than Class B. Also, if we have a look at the results of individual tasks, we can see that in the majority of tasks Class A produced better results. If we analyse the content of the test, the tasks that produce bigger deviation are related to the determination of the real and imaginary part of a complex number (TASK 4), complex conjugate of a complex number (TASK 5) and division of complex numbers (TASK 10). Tasks that produced similar results in both classes were related to the determination of the power of the imaginary unit (TASK 1), different forms of the presentation of a complex number (TASK 2) and addition and subtraction of complex numbers (TASK 8). Also, Figure 11. shows a graph with the results of on-line test made by Class A and Class B. Results of the Class A are marked in blue and the results of the Class B in red. Columns of the histogram are marked with the number of students who gave the correct answer and they are shown for each task in the test

5. Conclusions

Analysis of the students' achievements at the level of complex numbers in a "traditional" way and teaching enriched with multimedia software. The paper focuses on learning complex numbers using the GeoGebra/HotPotatoes multimedia software. This approach to teaching aims to increase student achievement, raise the teaching process to a higher level of efficiency, and motivate students to work and learn more. Students tested in Class A ("traditional" method) and Class B (GeoGebra/HotPotatoes multimedia software) have the following results: out of a possible 220 correct answers, students in Class A gave 166 (75.45%) and Class B 120 (54.54%) correct answers. We can conclude that Class A had a 20.91% better result than Class B. Also, if we look at the results of individual tasks, we see that in most tasks Class A achieved better results. If we analyses the content of the test, the tasks that produce a greater deviation refer to the determination of the real and imaginary part of a complex number (TASK 4), the complex conjugate of a complex number (TASK 5) and the division of complex numbers. (TASK 10). The tasks that obtained similar results in both classes were related to determining the potential of an imaginary unit (1st TASK), different forms of complex number representation (2nd TASK) and addition and subtraction of complex numbers (8th TASK).
Finally, we can conclude that the use of multimedia in teaching GeoGebra/HotPotatoes facilitates the process of knowledge transfer and enables students to actively participate, present their proposals for solving problems, research and gain self-confidence. With this approach to teaching, the teacher overcomes the limitations of classical teaching. The teacher offers a creative and interesting approach to teaching, and is obliged to design and adapt the teaching material to suit the students as best as possible. It is necessary to use performance tasks as much as possible, and to develop students' ability to apply different

techniques in solving these tasks. GeoGebra is a powerful tool that enables the fulfilment of all set goals in modern mathematics teaching, and is very easy to use by both teachers and students.

Competing interests

Authors/co-authors Dragana Nedić, Gordana Jotanović, Tijana Paunović and Aleksandar Kršić declare that there are no competing interests in the paper entitled "ANALYSIS OF STUDENT ACHIEVEMENTS IN TEACHING COMPLEX NUMBERS USING GEOGEBRA SOFTWARE".

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