

## SOLVING TASKS FROM LINEAR PROGRAMMING USING GEOGEBRA

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**Abstract.** Each student from every new generation, soon or latter, encounters difficulties when solving mathematical problems, even the best ones. Some students have problems with mastering the material, others with solving homework assignments, others have problems with quickly forgetting what they have learned, others think that they would lose a lot of time for solving the task and, in the end, they might get an incorrect result, which reduces their motivation for finding a solution and so on. Mathematical content that is presented graphically and mathematical problems whose solution may be obtained graphically remains best in the student's memory. Moreover, if an appropriate software is used during the solving of a given mathematical problem, if its graphical representation is precise enough and the final solution can be clearly seen from it, then most of the stumbling blocks for students will be surmounted. In this paper we will present the graphical solution method for linear programming problems using GeoGebra. The software allows graphic editing to be done in a quick and simple way which is very important for students.

### 1. INTRODUCTION

Whenever possible, a graphical representation that can be done in the fastest and most accurate way and its use to obtain a solution of a given mathematical problem enables a permanent memorization of what has been learned. Using an appropriate software for graphical solving will enable obtaining a solution in a much shorter time and perceiving the solution from the drawing itself. The graphic of the solved problem usually gives a complete picture of the solution which, for most of the students, is crucial for permanent memorization of what was perceived and learned. The best way to perform the graphical solution is with the use of an appropriate educational software. The software which we are going to use in this paper is GeoGebra. It is a free, open source, simple to use mathematical program that connects geometry, algebra, calculus, and statistics. The possibilities of GeoGebra as an educational software for mathematics are enormous. It can be used at all levels of education, from primary schools to universities, for drawing basic geometrical shapes to three dimensional objects, for performing basic analysis of functions with one variable to determining a conditional extremity of a function with two variables and visually presenting the conditional extremum (see for example [13]).

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One of the main features of GeoGebra is that it is a dynamic software. Unlike a sketch on paper, which is a static model, in GeoGebra it is possible to change certain parameters in the Graphics window simply by making changes in the Algebra window. GeoGebra's user interface is flexible and customizable as needed.

All about the main features of GeoGebra, along with its usage for solving mathematical problems from different mathematical topics, can be found in [2] and [6].

Several studies show that the use of educational software as well as other information technologies increases students' motivation, desire, and interest in solving problems. The main goal of the research in paper [12] is the question "Does the technical equipment of the classrooms bring better results in mastering the teaching program by the students?". The authors determine the quality of knowledge that the students get when learning the topic "Construction of triangle and quadrangle" with use of GeoGebra and informatics/mathematics approach, by comparing the achieved results on the diagnostic and the final test, of the experimental and the control group. The experimental group of students is learning the topic with use of software and constructions are made on computer, while the control group is learning by the traditional method of constructions made in notebook with ruler and compass.

In [1] authors analyze the perceptions and attitudes about the use of ICT tools for visualization as a "modern" approach for solving geometry problems in primary schools in Macedonia. In [15] the concept of a discrete random variable is introduced following the standard definitions, but by use of information technology, with emphasis on modeling probability situations with only two outcomes. Also, examples of discrete random variable with a geometric distribution are given, which are visually represented with GeoGebra.

The number of countries worldwide that have the development of the Information Society as one of their highest priorities has increased in the last decades. One of the key segments for the promotion and development of the Information Society is the education. The quality of the educational process is closely related to the application of the information and its communication technologies. In [16], the authors have presented the results of their research that has been conducted to investigate the factors that affect the motivation of teachers to use ICT in their classroom. There is more research in which the main goal is to see the importance of ICT in the teaching process in mathematical subjects. In [14] are given the results of the research which was conducted with students from two Universities: Mother Teresa, Skopje and Goce Delcev, Stip. Students were split into two groups. With one group, the mathematical content (algebra, geometry, analysis) was processed by using GeoGebra and on a computer, while with the other group the same material was processed without any kind of visualization. After that the testing was done. The comparison of the results led to the conclusion that the visualization of problems, the inclusion of visualization software in the curriculum, introducing students to the importance

of mathematics and its extensive application is very important to do during the educational process.

For increasing the motivation for learning mathematics and increasing the level of knowledge, a web application <http://mathlabyrinth.azurewebsites.net> is presented in [11]. This application is for students in the secondary education and contains mathematical problems that are related to real-life problems the students may encounter.

Many high school teachers face questions from their students about the applicability of the mathematical contents. In [3], the authors address students' questions related to linear programming problems and solved them using GeoGebra. The article [9] describes the observations of the experimental teaching conducted in the high school in Košice, where GeoGebra was used. GeoGebra was used for the first time in students' lives for solving a linear optimization word problem. The findings in [5] show that the use of GeoGebra enhanced the students' performance in learning linear programming, hence it was recommended that teachers employ GeoGebra software in teaching and learning Linear Programming and any other mathematics topics. The study [10] aims to determine the increase in students' critical thinking skills in linear programming learning through the Problem Based Learning (PBL) model assisted by GeoGebra Software. This research is a semi-experimental study with one group pre-test post-test design. The group in this study involved 24 students. The instrument used was the pre-test and post-test questions on critical thinking skills. The data were analyzed using SPSS software. The results showed that the use of the PBL model assisted by GeoGebra software can improve students' critical thinking skills on linear programming material.

## 2. INITIAL RESULTS

In this section we will graphically solve a few linear programming problems (LP problems) using GeoGebra. Then we will look at the results of the survey carried out with a group of 20 students from the Faculty of computer science at University Goce Delcev Stip, which consists of answering a questionnaire whose questions are related to the advantages and disadvantages of using educational software when graphically solving linear programming problems.

Through the solved examples, we will show few ways to reach the solution with the help of GeoGebra. The examples are given below.

**Problem 1.** Find graphically the maximum and the minimum of the function  $L = x + y$ , with the following constraints:

$$\begin{cases} -x + y \geq -3 \\ x \leq 4 \\ x + 2y \leq 10 \\ -x + y \leq 2 \\ x \geq 0 \\ y \geq 0 \end{cases} .$$

**Solution:** First step is to draw the bounding lines of the constraints. In the Input bar of GeoGebra we write, one by one, the following equations:

$$\begin{aligned} -x + y &= -3, \\ x &= 4, \\ x + 2y &= 10, \\ -x + y &= 2, \\ x &= 0, \\ y &= 0. \end{aligned} \tag{1}$$

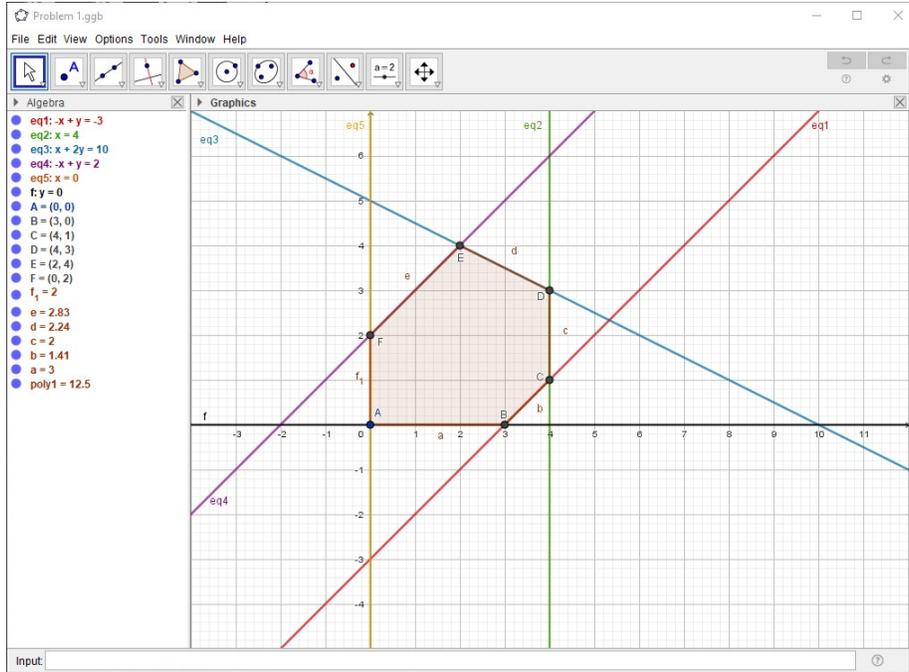
As we enter each equation, it will automatically appear in the Algebra window and the corresponding line will be drawn in the Graphics window. Each line can be represented by a different colour. The colour of the equation in the Algebra window will be the same as the colour of the corresponding line in the Graphics window.

The lines in (1) divide the plane into few regions, only one of which is the feasible region. We can discard the regions that are left from the  $y$  axis (any point, in any of these regions, does not satisfy the constrain  $x \geq 0$ ), those right from the line  $x = 4$  and those below the  $x$  axis. Then, using the one-point-test, we check which one of the remaining regions (five in total) satisfies all the given constrains. This is the polygon with the vertices in the intersection points of the lines with equations:

- $x = 0$  and  $y = 0$  (i.e., the origin),
- $y = 0$  and  $-x + y = -3$ ,
- $-x + y = -3$  and  $x = 4$ ,
- $x = 4$  and  $x + 2y = 10$ ,
- $x + 2y = 10$  and  $-x + y = 2$ , and
- $-x + y = 2$  and  $x = 0$ .

We find these points using the Intersect tool of two objects  and then we connect them with the Polygon tool . The result is given in Figure 1.

If we equate the objective function to 0 we'll get  $x + y = 0$ , i.e.,  $y = -x$ , we may perceive the objective function as the line  $y = -x$  "moving" from left to right as its value increases (Really?).



**Figure 1:** The feasible region for Problem 1 (polygon  $ABCDEF$ )

So, the objective function will reach its maximum when the line  $y = -x$  “passes” through the “last” vertex of the feasible region while increasing its distance from the origin. In this case that will be the point  $D(4,3)$ . Hence  $\max L = 7$ .

Similarly, if we are looking for the minimum, it will be reached when the line passes through the origin  $(0,0)$  and hence,  $\min L = 0$ . ■

**Remark 2.** If the objective function is of form

$$L = ax + yb,$$

most of the teachers and textbooks, suggest that direction in which the objective function increases should be represented by two or more lines obtained by assigning increasing values to the objective function. These lines are usually referred as isoprofit lines (for maximization LP problems), isocost lines (for minimization LP problems), or as “objective function lines”. Usually they are also drawn on the graph, as shown on Figure 2. This can be quite helpful. From Figure 2, we can easily conclude that the objective function has a maximum at point  $D(4,3)$  and that  $\max L = 7$ .

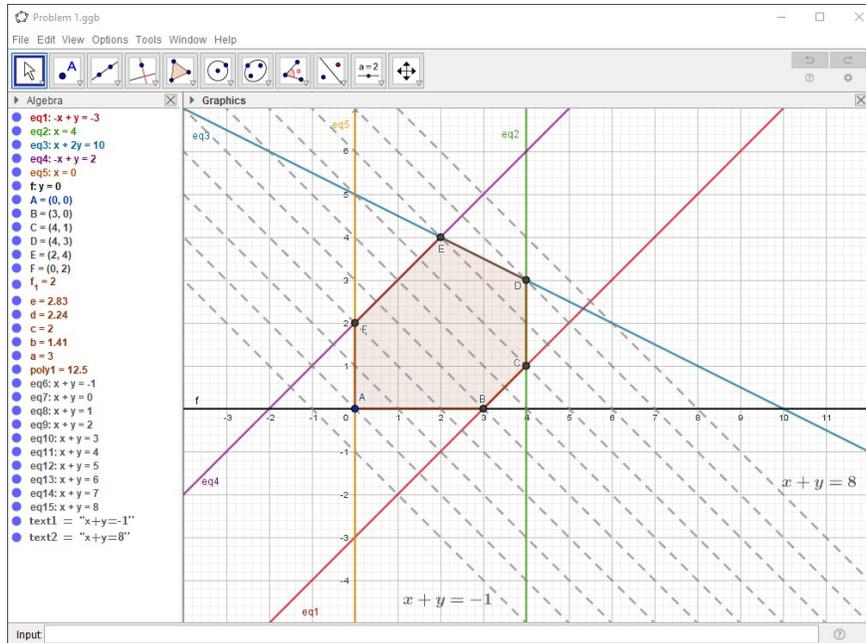


Figure 2: Objective function lines from Problem 1

**Problem 3.** Find graphically the smallest and largest value of the function  $L = x + y$  on the area:

$$\begin{cases} 3x + 2y \geq 6 \\ x - 2y \leq 2 \\ -3x + 2y \leq 6 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

**Solution:** As in the previous problem, in the Input bar we write the equations of the bounding lines, one by one,

$$3x + 2y = 6, \quad x - 2y = 2, \quad -3x + 2y = 6, \quad x = 0 \quad \text{and} \quad y = 0.$$

For this problem we will obtain the feasible region with GeoGebra in a different way. GeoGebra supports graphical representation of a single inequality with two variables and a system of inequalities with two variables as well. For the system of inequalities given in the problem, in the Input bar we write:

$$3x + 2y \geq 6 \wedge x - 2y \leq 2 \wedge -3x + 2y \leq 6 \wedge x \geq 0 \wedge y \geq 0.$$

The symbols  $\leq, \geq, \wedge$  are contained in the palette available by clicking on  at the end of the Input bar.

The result obtained with GeoGebra is given in Figure 3. The graphic suggests that the feasible region may be unbounded (but not necessarily!). If, after few clicks with the Zoom out tool we don't get a bounded region, chances are that the region is indeed unbounded. But we need to verify this algebraically. By Figure 3, the feasible region may contain every point of the line  $y = x$  which satisfies  $x \geq 2$  i.e., the whole ray  $R = \{(x, y) : x \geq 2, y = x\}$ . It can be easily verified that this is indeed true: all the constraints in the problem are satisfied for every point in  $R$ . Since the ray is unbounded, the feasibility region will also be unbounded.

As in the solution of Problem 1, from Figure 3 we can see that the objective function reaches its smallest value  $L = 2$  for  $x = 2, y = 0$ . But the largest value does not exist (there is no "last" point in the feasible region through which the line  $y = -x$  "passes" while "moving" from left to right). ■

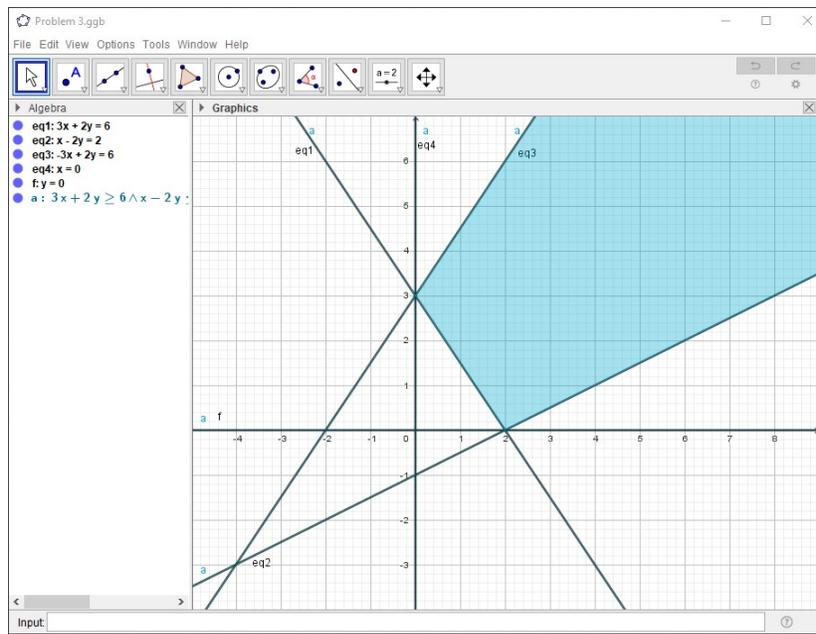


Figure 3: The bounding lines and the feasible region from Problem 3

**Problem 4.** Find graphically the minimum of the function  $L = 2x + 3y$  with constraints:

$$\begin{cases} x + y \geq 2 \\ x + 3y \leq 12 \\ 3x + y \leq 12 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

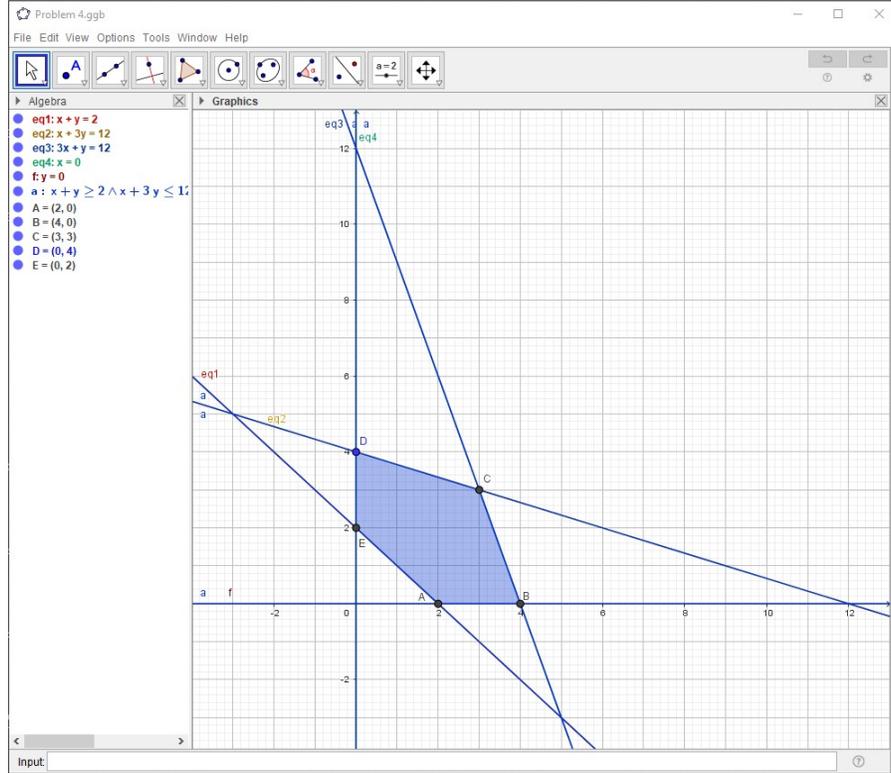
**Solution:** In the Input bar, one by one, we enter the equations of the bounding lines

$$x + y = 2, \quad x + 3y = 12, \quad 3x + y = 12, \quad x = 0 \quad \text{and} \quad y = 0,$$

and then

$$x + y \geq 2 \wedge x + 3y \leq 12 \wedge 3x + y \leq 12 \wedge x \geq 0 \wedge y \geq 0.$$

Once the feasibility region is visible in the graphics view, using the intersection tool we find its vertices. The result in GeoGebra is given in Figure 4.



**Figure 4:** The feasible region from Problem 4

We check the value of the function  $L = 2x + 3y$  at each of the points  $A(2,0)$ ,  $B(4,0)$ ,  $C(3,3)$ ,  $D(0,4)$  and  $E(0,2)$  by entering in the Input bar, one by one, the following expressions:

$$2x(A) + 3y(A),$$

$$2x(B) + 3y(B),$$

$$2x(C) + 3y(C),$$

$$2x(D) + 3y(D),$$

$$2x(E) + 3y(E).$$

The values will be displayed in the Algebra window. So, we get:

$$\begin{aligned} 2x(A) + 3y(A) &= 4, \\ 2x(B) + 3y(B) &= 8, \\ 2x(C) + 3y(C) &= 15, \\ 2x(D) + 3y(D) &= 12, \\ 2x(E) + 3y(E) &= 6. \end{aligned}$$

Hence, the minimum value is  $L = 4$  which is obtained when  $x = 2, y = 0$ . ■

Next, we will give an example of a word problem.

**Problem 5.** Products of a same kind are manufactured in two production plants A and B. 250 units are produced in plant A and 350 units in plant B. Three stores I, II and III have a demand of 150, 240 and 210 units, respectively. The transportation costs per unit from the production plants to the stores are given in the following table:

**Table 1:** Transportation costs per unit

<i>Plant</i> \ <i>Store</i>	I	II	III
A	4	3	5
B	5	6	4

Find a transportation plan so that the total transportation expenses will be the lowest.

**Solution:** Since the given problem is a balanced transportation problem (i.e., the total amount of units produced in both plants A and B equals the total amount of demanded units by all three stores) we may proceed as follows. Let  $x$  denote the number of products transported from plant A to store I and  $y$  the number of products transported from plant A to store II. Since the needs of store I are 150 units, it is necessary to bring  $(150 - x)$  units from B to I. Also, from plant B to store II it is necessary to bring  $(240 - y)$  units. Further, plant A produces 250 units and we have already allocated  $(x + y)$  units. This means that from A to III will be transported  $(250 - x - y)$  units. To ensure that the demand of store III is met, the number of units that we need to transport from B to III is  $210 - (250 - x - y) = x + y - 40$ . Thus, we have the following transportation plan:

**Table 2:** Transportation plan

<i>Plant</i> \ <i>Store</i>	I	II	III
A	$x$	$y$	$250 - x - y$
B	$150 - x$	$240 - y$	$x + y - 40$

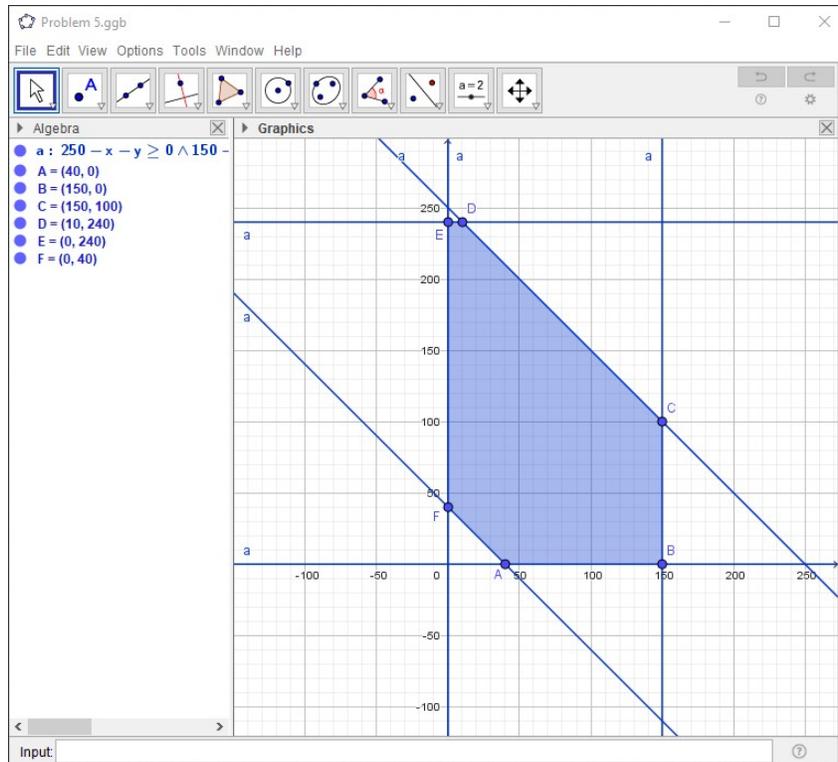
By Table 1, the total transportation expenses will be:

$$\begin{aligned} L &= 4x + 3y + 5(250 - x - y) + 5(150 - x) + 6(240 - y) + 4(x + y - 40) \\ &= -2x - 4y + 3280. \end{aligned} \quad (2)$$

According to the condition of the problem, it is necessary to find the minimum of the function in (2). But in this example  $x$  and  $y$  cannot take arbitrary values. The quantity of products cannot be a negative number. Therefore, all the numbers in table 3 are nonnegative, that is:

$$\begin{aligned} x \geq 0 \wedge y \geq 0 \wedge 250 - x - y \geq 0 \wedge 150 - x \geq 0 \wedge \\ 240 - y \geq 0 \wedge x + y - 40 \geq 0 \end{aligned} \quad (3)$$

which means we need to find a minimum of the function in the area given by the system of inequalities (3). That area is shown in Figure 5.



**Figure 5:** The feasible region from Problem 5

The area is obtained by entering the inequalities in the input field in the GeoGebra window that is in the input field we enter:

$$x \geq 0 \wedge y \geq 0 \wedge 250 - x - y \geq 0 \wedge 150 - x \geq 0 \wedge 240 - y \geq 0 \wedge x + y - 40 \geq 0.$$

The vertices of the polygon are the points:

$$A(40,0), B(150,0), C(150,100), D(10,240), E(0,240), F(0,40),$$

The function will take the smallest value at one of the vertices of the polygon  $ABCDEF$ . We check the value of the function in each of the points  $A, B, C, D, E, F$  by entering in the Input bar, one by one the following expressions:

$$\begin{aligned} & -2x(A) - 4y(A) + 3280, \\ & -2x(B) - 4y(B) + 3280, \\ & -2x(C) - 4y(C) + 3280, \\ & -2x(D) - 4y(D) + 3280, \\ & -2x(E) - 4y(E) + 3280, \\ & -2x(F) - 4y(F) + 3280. \end{aligned}$$

In the Algebra window we obtain:

$$\begin{aligned} & -2x(A) - 4y(A) + 3280 = 3200, \\ & -2x(B) - 4y(B) + 3280 = 2980, \\ & -2x(C) - 4y(C) + 3280 = 2580, \\ & -2x(D) - 4y(D) + 3280 = 2300, \\ & -2x(E) - 4y(E) + 3280 = 2320, \\ & -2x(F) - 4y(F) + 3280 = 3120. \end{aligned}$$

The minimum value is obtained at point  $D(10,240)$ . Hence, for the transportation plan we get the following table:

**Table 3:** Transportation plan

<i>Store</i> <i>Plant</i>	I	II	III
A	10	240	0
B	140	0	210

which means that:

- 10 units of product should be transported from plant A to store I,
- 240 units of product should be transported from point A to point II, etc.

For this transportation plan the value of the total expenses will be 2300. ■

In order to confirm our opinion and firm belief for the benefits of the use of software for learning mathematics, as well solving mathematical problems, especially when applying the graphical solution method, we created a group of 20 students from the Faculty of computer science who voluntarily attended classes where they learned how to use the GeoGebra software while learning different mathematical topics. In these classes, in addition to other mathematical

topics, Linear programming was also covered. After the classes, at the next meeting, the students received a questionnaire that they had to answer in a short time. Questionnaire and the results are given on the next page.

Students completed the questionnaire in a very short time. All 20 students answered “yes” to five from ten questions. For the rest five questions most of the students also answered “yes”, and just few with “no” or “maybe”. The responses could only confirm our opinion that students benefit greatly from using the software. The results from a questionnaire are given in Table 4 and Figure 6.

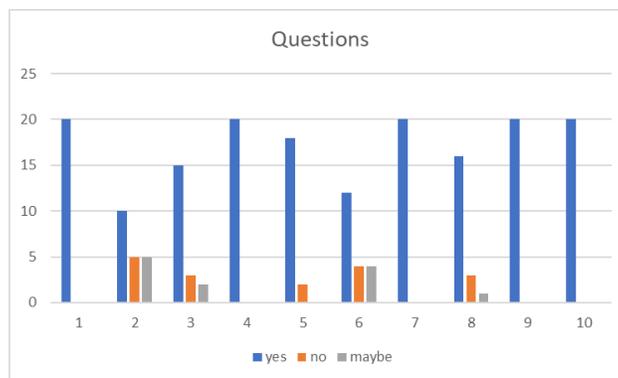
### QUESTIONNAIRE

Question	Answer
1. With help of GeoGebra software solving tasks of linear programming is taking less time than manually?	Yes/ no / maybe
2. Did the GeoGebra software help you to permanently remember the solved tasks in your memory?	Yes/ no / maybe
3. When solving linear programming tasks graphically at home, would you use GeoGebra software?	Yes/ no / maybe
4. Does the quick and accurate solution of problems from linear programming graphically with the help of GeoGebra software increase the motivation of students for learning problems from this topic and for learning mathematical problems in general?	Yes/ no / maybe
5. Does GeoGebra educational software is a good choice for solving graphical linear programming tasks?	Yes/ no / maybe
6. Would you continue to follow additional classes in which tasks from various mathematical topics would be solved with the help of GeoGebra software?	Yes/ no / maybe
7. Do you think that your success in math subjects will be higher after using GeoGebra software for problems solving?	Yes/ no / maybe
8. Is it interesting for you solving problems of linear programming graphically using GeoGebra software?	Yes/ no / maybe
9. Would you recommend to other students using the software when learning any math topic?	Yes/ no / maybe
10. Do you think there should be books that explain how to solve graphical problems from linear programming using GeoGebra software?	Yes/ no / maybe

Answers:

**Table 4:** Results from a questionnaire

Question (#)	yes	no	maybe
1	20	0	0
2	10	5	5
3	15	3	2
4	20	0	0
5	18	2	0
6	12	4	4
7	20	0	0
8	16	3	1
9	20	0	0
10	20	0	0



**Figure 6:** Results from a questionnaire

### 3. FEW ADDITIONAL EXAMPLES

In this section, through few additional examples, we are going to give an updated version of the instructions contained in [8] on how the use GeoGebra for the graphical solution of a given LP problem. Our goal is to take advantage, as much as possible, of GeoGebra's powerful tools to obtain fast and accurate graphical solution for a given LP problem with two decision variables.

The algebraic and the graphical method for solving LP problems with two variables are usually performed simultaneously. Sometimes we can obtain the solution only with one of the methods, but sometimes each method, if performed alone may lead to an incorrect answer. So, we always encourage students to do some blend of these methods.

In general, the algebraic method consists of finding the coordinates of the intersecting points of the bounding lines as solutions of systems of two linear equations with two variables. We have to consider all the systems that we can

form from the equations of the bounding lines. Then, we check which of the points that are a unique solution to some of these systems are extreme points (i.e., satisfy all the given constraints – for the LP problem we can immediately discard all points for which we get at least one negative coordinate). Finally, we calculate the values of the objective function at the extreme points and determine which of these values is the smallest and which is largest. But this method can be employed only when the feasible region is bounded. To verify algebraically (only) that the feasible region is indeed bounded usually takes much more time than, for example, drawing a quick sketch on paper or use GeoGebra to draw a *convex* polygon with vertices at the extreme points. Otherwise, we may obtain an incorrect answer. For example, if we use only the algebraic method to get the solution of Problem 3, then we will have to find the solution of  $C_5^2 = \frac{5!}{2!3!} = 10$  systems of two linear equations with two variables.

Each of these systems will have a unique solution. Three of the systems will have the same solution  $x = 2$ ,  $y = 0$ , i.e., the point  $A(2,0)$ , which is an extreme point, another three will have the same solution  $x = 0$ ,  $y = 3$ , i.e., the point  $B(0,3)$ , which is also an extreme point, while the solutions of the other four systems are not extreme points. Since it is not unusual to obtain that the feasible region is a segment in the plane, which is a bounded convex set, some students proceed with calculating the values of the objective function and conclude that  $L = x + y$  has a minimum  $L = 2$  at  $A(2,0)$  and maximum  $L = 3$  at  $B(0,3)$ .

On the other hand, the accuracy of the solution obtained with the graphical method mainly depends on:

- the accuracy of the graphical representation of the feasibility region,
- the accurate determination of the points at which the objective function has a minimum and/or maximum (usually done by drawing two or more objective function lines).

So, most of the time, if we want to get a correct answer or verify our conclusions from the graphics, we usually calculate the coordinates of the points at which the minimum or/and maximum is attained by solving systems of equations, and after that we calculate the value of the objective function at those points.

The general steps of the graphical method, regardless of whether it is performed on paper or with the help of some software, were covered with the examples in the previous section. The way GeoGebra is used in the solution of Problem 1, is the one that most closely resembles the way we usually perform the graphical solution on paper. We deliberately left one of the most common misconceptions that is found both among teachers and students, and in some textbooks as well. It is about the geometrical interpretation of the objective function. Although this misconception rarely affects the accuracy of the final

solution, since it is one of the crucial elements for building a more effective GeoGebra file, we will address this first.

In the solution of Problem 1 we've placed the question "Really?" in small brackets. The question is not about the interpretation of the objective function as a movable line across the plane, but the direction in which it moves.

The objective function of a given LP problem with two variables is a linear function (with two variables) of form:

$$f = f(x, y) = ax + by,$$

where  $a, b \in \mathbb{R} \setminus \{0\}$  are given numbers. When we draw the objective function lines (the isoprofit lines or the isocost lines) on the graph, we are actually trying to determine which of the elements from the one parameter family of parallel lines:

$$\{ax + by = f : f \in \mathbb{R}\}, \tag{4}$$

intersects the feasible region (see for example Ch. 23, Example 5.1 in [7]). Since the lines in (4) may be regarded as different positions of a line that moves across the plane, the geometrical interpretation of the objective function as a movable line on the plane is perfectly fine and quite useful when we perform a graphical solution of a given LP problem. But, as the values of objective function increase, we would get the same set of dashed lines on Figure 2 if we perceive the movement of the line from top to bottom, or from southwest to northeast, or in the direction of *any* vector  $\vec{v} = (x, y)$  whose coordinates satisfy  $x + y \geq 0$ . This would indicate that the movement of the line can be interpreted as the "sliding across the plane". However, the direction in which the line moves is not always from left to right, or from bottom to top etc. Hence, the movement is not sliding. The direction in which the line moves is not arbitrary as we usually think or say. It moves:

- in the direction of the vector  $\vec{n}_f = (a, b)$ , as  $f$  increases,
- in the opposite direction of the vector  $\vec{n}_f = (a, b)$ , as  $f$  decreases.

The vector  $\vec{n}_f = (a, b)$  is perpendicular to every line in (4) and is called the *direction* (or the *gradient*) of the objective function (see for example Ch. 3, Section 3.1 in [4]). So, the line  $x + y = 0$  (or  $y = -x$ ) in Problem 1 and Problem 2, actually does not move from left to right as the value of the objective function increases, it moves in the direction of the vector  $\vec{n}_L = (1, 1)$ .

Instead of "sliding across the plane", a more accurate analogy would be a "rolling line" (like the rolling of a cylindrical straight pencil which is initially placed parallel to the objective function lines, the pencil may slide in any direction, and still remain parallel to the objective function lines, but it can roll in only two directions to remain parallel to these lines). Thus, instead of drawing a net of objective function lines, we may simply draw the vector  $\vec{n}_f = (a, b)$  and, if necessary, lines that are perpendicular to this vector trough

the extreme points that may be the points at which the objective function has minimum and/or maximum.

Now let's see some additional examples.

**Problem 6.** Find the minimum and the maximum of the function  $f = 2x + y$  over  $X$  if,

$$X: \begin{cases} -2x + y \leq 1 \\ x + 2y \leq 9 \\ x \leq 5 \\ y \leq 3 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

**Solution:** *Step 1: Determining the feasibility region.* In the Input bar we enter the following expressions (commands), one line at a time:

```

b1:= -2x+y=1
b2:= x+2y=9
b3:= x=5
b4:= y=3
b5:= x=0
b6:= y=0
n1:= -2x+y<=1
n2:= x+2y<=9
n3:= x<=5
n4:= y<=3
n5:= x>=0
n6:= y>=0
X:= n1 && n2 && n3 && n4 && n5 && n6
    
```

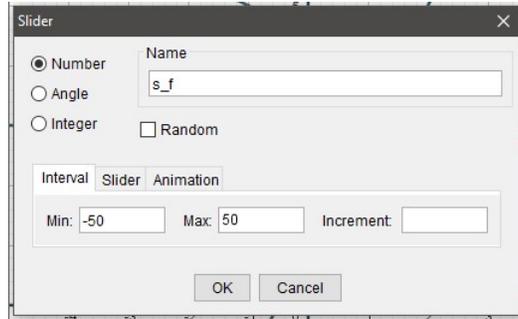
In this way we will assign more meaningful names for the objects instead of the letting GeoGebra to do the naming. After entering these expressions, the Graphic window will be quite cluttered, so we need to turn of the visibility of the individual inequalities n1, n2, n3, n4, n5 and n6. The commands for the symbols  $\leq, \geq, \wedge$  which were used in the previous section are `<=`, `>=`, `&&` respectively. After completing this, from the graphic obtained in the Graphics window, we can see that the feasible region (obtained with the last command) is bounded and determine which pairs of the lines b1, b2, b3, b4, b5 and b6 have an intersection at a vertex of the feasible region. Then, either by using the Intersect tool, or entering in the Input bar the following commands, one line at a time:

```

A:=Intersect(b5,b6)
B:=Intersect(b6,b3)
C:=Intersect(b3,b2)
D:=Intersect(b2,b4)
E:=Intersect(b4,b1)
F:=Intersect(b1,b5)
    
```

we determine the coordinates of vertices of the feasible region.

*Step 2: Representing the objective function as a movable line.* First, with the Slider tool  we create a slider in the Graphics window. A window as in Figure 8 will appear. We'll change the default name from "a" to, let's say "s\_f", which will give the name  $s_f$  to the slider in the Algebra window. We'll replace the values -5 and 5 with "larger" ones. Although the coefficients of the objective function and the constraints are relatively small (by absolute values), for cases like this, in our experience the interval for the slider may go from -50 to 50 (and even wider). If this doesn't work well, we can adjust these values later. At this moment will leave the Increment empty (i.e., at its default value 0.1).



**Figure 7.** Initial settings for the slider

Next, in the Input bar we enter the following expressions, one line at a time:

```
n_f:= (2,1)
m_f:= x(n_f)*x+y(n_f)*y=s_f
f(x,y)= x(n_f)*x+y(n_f)*y
```

The first expression is the direction of the objective function. Sometimes, to ensure that the direction is visible in the Graphics window (not too small, not too large) we need add to another vector, like

```
p_f:= q*n_f
```

for an appropriate choice of a *positive* value for  $q$  (see the Problem 8.b). The second expression is the line that will visualize the "movement" of the objective function line in the Graphics window. This line will move:

- in the direction of the objective function, as the value the slider increase,
- opposite of the direction of the objective function, as the value the slider decrease,

while the third expression is the actual objective function. We could've entered  $m_f:= 2*x+y=s_f$  and  $f(x,y)= 2*x+y$ , but we want to use the dynamic property of GeoGebra as much as possible (see Remark 7).

Before we proceed, we need to test the slider. For a bounded feasible region, the line  $m_f$  should move so that it can pass through and exit the feasible region as we drag the dot on the slider in both directions, and yet, remain visible in the

Graphics window (assuming that its visibility is not turned off in the Algebra window). If the movable line  $m_f$  is still intersecting the feasible region or not visible at all, either when the slider is at its minimum or at its maximum value, then we should replace the values -50 to 50. For this problem, we may adjust the slider from -5 to 20.

After Step 1 and Step 2, by changing the colour of the vector  $n_f$ , the slider  $s_f$  and the movable line  $m_f$ , we'll get something like Figure 8.

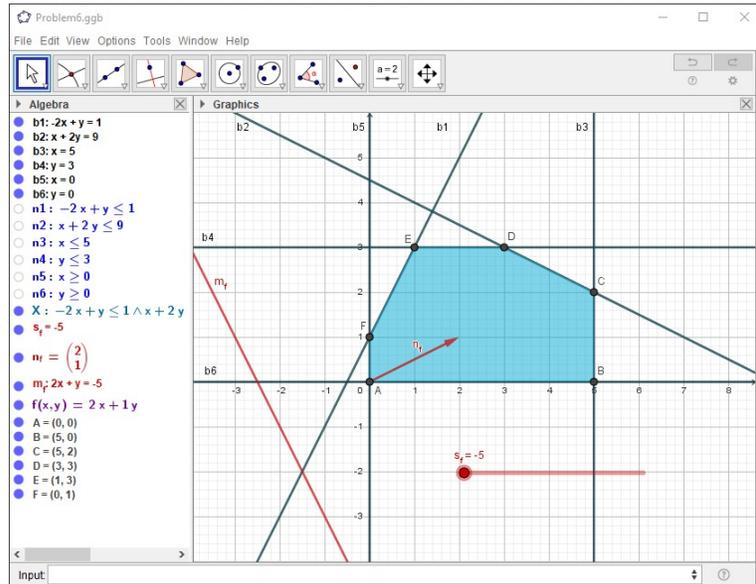


Figure 8: Initial elements for the graphical solution

Next, we need to slowly drag the dot on the slider and pay attention to the values above the slider (or those of  $s_f$  and on the right side of the equation in  $m_f$  in the Algebra window). As we drag the dot from lowest to highest value of the slider, the order of the vertices of  $X$  in which the movable line  $m_f$  in Graphics window will pass through are:

$$A (\approx 0), F (\approx 1), E (\approx 5), D (\approx 9), D (\approx 10) \text{ and } C (\approx 12).$$

The values in the parenthesis are the approximate values that will appear above the slider.

**Note:** Since the coordinates of these points and the coefficient of the objective function are all integers, we may obtain the exact values while dragging the dot on the slider if we set 1 as the Increment value of the slider.

At this point, having in mind that a linear function will attain its minimum and maximum on a bounded convex polyhedral set at its extreme points, we already have all the information that we need to give an answer to the problem. The objective function attains its:

- minimum at  $A(0,0)$  and  $\min_X f = 2 \cdot 0 + 0 = 0$ ,

- maximum at  $C(5,2)$  and  $\max_X f = 2 \cdot 5 + 2 = 12$ .

But, let the GeoGebra confirm the results. In the Input bar we enter the following expressions, one line at a time:

```
ValueAtA:=f(A)
ValueAtB:=f(B)
ValueAtC:=f(C)
ValueAtD:=f(D)
ValueAtE:=f(E)
ValueAtF:=f(F)
```

The Algebra window will be updated with the corresponding objects as shown in Figure 9. ■

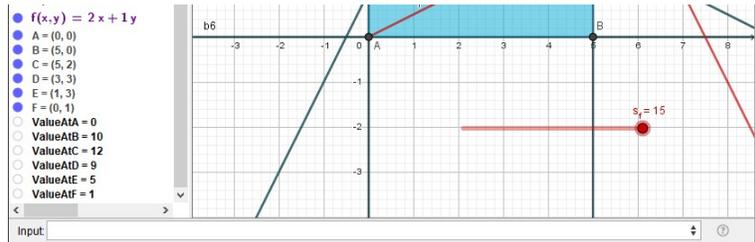


Figure 9: Values of the objective function at the extreme points

**Remark 7:** After entering the expressions for the bounding lines, the inequalities for the constraints and the feasible region, we'll get the same results as in Section 2. The advantages of the approach described in the solution of Problem 6 are, at least, two folded. First, we may easily encounter LP problems, especially when working with real-life problems, without any feasible solution. This will occur whenever two or more of half planes determined with the constraints don't intersect. For example, if:

$$X : \begin{cases} x + y \leq 4 \\ x + y \geq 8 \\ x \geq 0 \\ y \geq 0 \end{cases},$$

and if we directly enter in the Input bar

$$x + y \leq 4 \wedge x + y \geq 8 \wedge x \geq 0 \wedge y \geq 0,$$

then nothing will be displayed in this Graphics window, which will not be the case if we first enter separately the inequalities of the constraints (and then the appropriate command for their intersection like the one for the set  $X$  in the solution of Problem 6).

Second, we want to use the dynamic feature of GeoGebra as much as possible. We can easily make any changes or corrections if needed. Small changes in the coefficients of the objective function, or the constraints, or

replacing one or more of the inequality signs with the reverse ones, may have huge impact on the solution of the problem. Let's see how it works.

We are going to use the GeoGebra file created for Problem 6 without the suggested modification of the Increment of the slider. We've saved this file under the name Problem6.ggb, made two copies of it and renamed them as:

Problem6(modified for Problem8a).ggb, and  
Problem6(modified for ForProblem8b).ggb.

**Problem 8.** a) Find the maximum of the objective function in Problem 6 over the sets

$$Y: \begin{cases} -2x + y \leq 1 \\ 3x + y \leq 17 \\ x \leq 5 \\ y \leq 3 \\ x \geq 0 \\ y \geq 0 \end{cases}, \quad \text{and} \quad Z: \begin{cases} -2x + y \leq 1 \\ x \leq 5 \\ y \leq 3 \\ x \geq 0 \\ y \geq 0 \end{cases}.$$

b) Find the maximum of  $f = 5x + 10y$  over the set  $X$  given in Problem 6.

**Solution: a)** Each of the sets of constrains  $Y$  and  $Z$  can be derived from  $X$ :  $Y$  by replacing the second constrain in  $X$  with  $3x + y \leq 17$ ,  $Z$  by completely removing the second constrain in  $X$ .

Now let's open the file Problem6(modified for Problem8a).ggb and continue where we left in the solution of Problem 6.

In the Algebra window, first by double clicking on the object b1 we will adjust the coefficients of  $x + 2y = 9$  to obtain  $3x + y = 17$ , then by double clicking on the object n1, we will adjust the coefficients of  $x + 2y \leq 9$  to obtain  $3x + y \leq 17$ .

And that is all.

At least for now.

As we can see from Figure 10, everything is adjusted at once. But now we have something different. From the changes in the last six objects in the Algebra window, which are made automatically, we see that the maximum of the function is no longer at  $C(5,2)$ , but in  $D$ . Moreover, due to the rounding, the first coordinate of  $D$  is not the exact one (from the information displayed in the Algebra window the coordinates of this point are  $D(4.67,3)$ ). The maximum value of the objective function is also not the exact one, but an approximate value of it (12.33). If we want to get the exact coordinates of  $D$  and the exact value of the maximum, we need to solve the system of two linear equations formed by the equalities b2 and b4. We can do this on paper or employ GeoGebra do this for us. GeoGebra has two commands for solving systems of linear equations, `Solve` and `Solutions`, but they are available only through the CAS module. We will use the first command. So, let's open the CAS window.

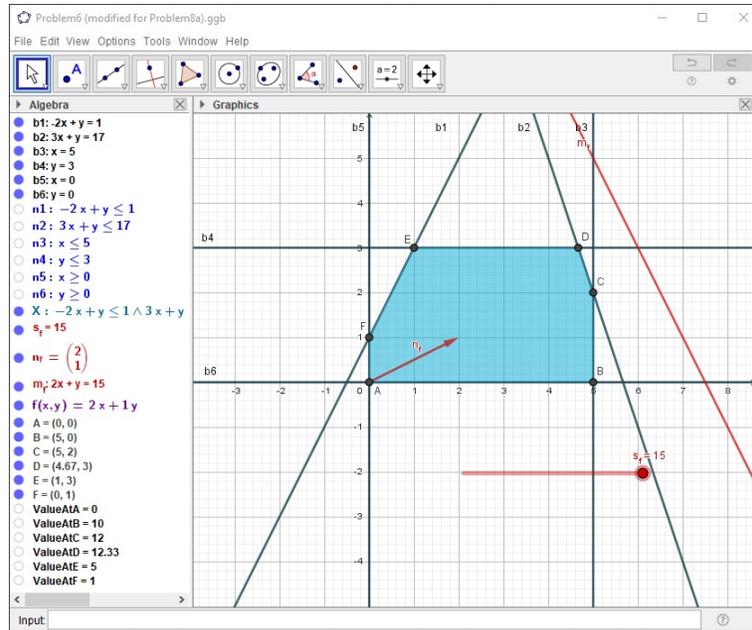


Figure 10: The results after changing one constrain from Problem 6

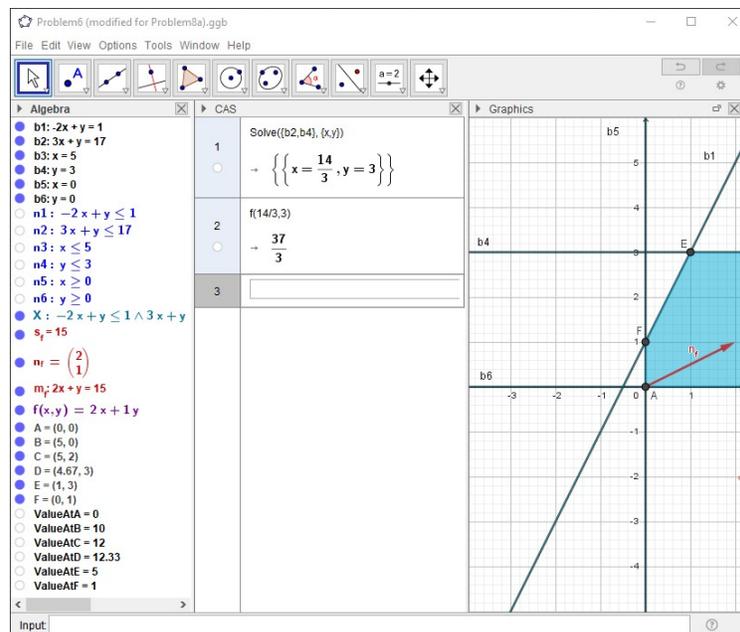


Figure 11: Use of the CAS module in GeoGebra for Problem 8.a).

Since we already have all the equations of the bounding lines, and we've obtained the point  $D$  as in intersection point of the lines  $b2$  and  $b4$ , in the Input bar *in the CAS window* (see Figure 11) we enter:

$\text{Solve}(\{b2, b4\}, \{x, y\})$

The result will be displayed bellow in form of two equations,  $\left\{ \left\{ x = \frac{14}{3}, y = 3 \right\} \right\}$ .

No rounding of the first coordinate! Now that we have the exact coordinates of  $D$ , since the Algebraic windows already contains an expression for calculating the values of the objective function, in the next Input bar *in the CAS window* we enter:

$f(14/3, 3)$

The result displayed bellow will be  $\frac{37}{3}$ . Hence,  $f = 2x + y$  has a maximum value  $\frac{37}{3}$  at point  $D\left(\frac{14}{3}, 3\right)$ .

Now let's find the maximum of  $f = 2x + y$  over  $Z$ . We can save the above changes, close the file, and work on a copy of it (if we want to preserve the changes), or continue and lose all the results we've just obtained for the set  $Y$ .

For the set  $Z$  we must proceed cautiously. The best is to follow these steps in exact order, except maybe interchanging step 3 and 4. All actions are on the objects in the Algebra window.

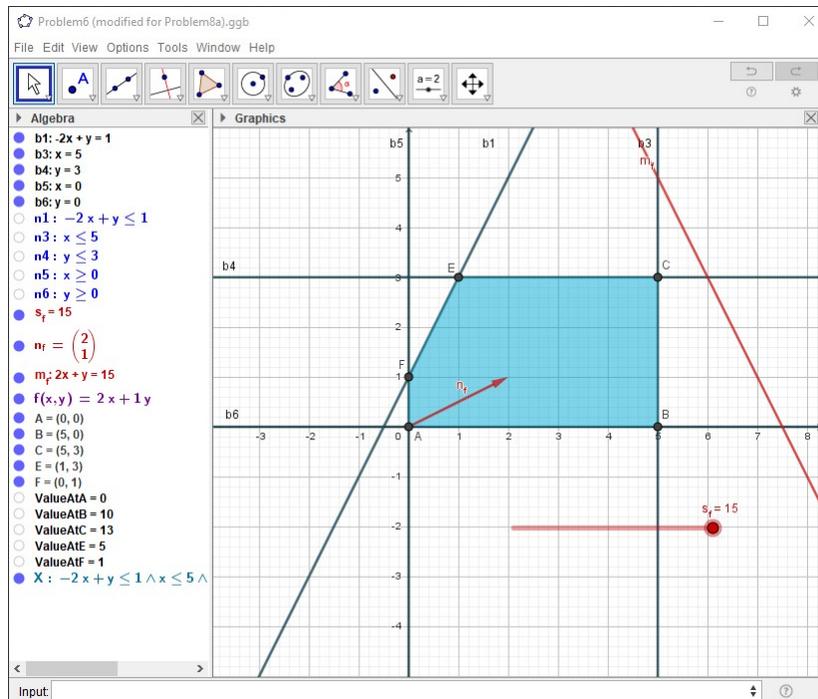


Figure 12: Result in GeoGebra after changes for the set  $Z$  in Problem 8.a)

1. double click on  $X$  and in the new window delete:  $\wedge n_2(x, y)$ ,
2. delete the inequality  $n_2$ ,
3. double click on  $C$ , and the new window replace  $b_2$  with  $b_4$ ,
4. double click on  $D$ , and the new window replace  $b_2$  with  $b_3$ ,
5. delete one of the points  $C$  or  $D$ ,
6. delete  $b_2$ .

The result should look the image in Figure 12. This time we have: the maximum of  $f = 2x + y$  over  $Z$  is 13, and it's attained at point  $C(5,3)$ .

**b)** Now we'll work on file Problem6(modified for ForProblem8b).ggb. We need to find the maximum value of the function  $f = 5x + 10y$  over the set  $X$  defined in Problem 6. For this, the only change that we have to make is on direction of the objective function  $n_f$  and observe how everything adjusts, both in Algebraic and in Graphics windows after that. By double clicking on the vector  $n_f$  in the Algebra window we replace its coordinates  $(2,1)$  with  $(5,10)$ . This vector is quite large in "size" relative to the other objects, but we must not make any further changes, since this will have effects on the values of the objective function. Instead, we'll turn off the visibility of the vector  $n_f$  and, to keep the direction of the movement of the line  $m_f$  in the Graphical windows, we enter in the Input bar:

$$p_f := 0.5 * n_f$$

and then, if necessary, change its colour and line style.

That's all.

At least for now.

The result should look like the image in Figure 13.

Now look at the results we obtained. There are two different points at which we have largest value of 45 of the objective function:  $C(5,2)$  and  $D(3,3)$ . Both points lie on a same objective function line:

$$5x + 10y = 45.$$

For every point on this line, hence for every point of the segment with end points at  $C$  and  $D$  (which is included in the feasible region), the value of the objective function is 45.

So, the maximum of  $f = 5x + 10y$  over  $X$  is attained at infinitely many points, each point on the segment with end points at  $C$  and  $D$ . ■

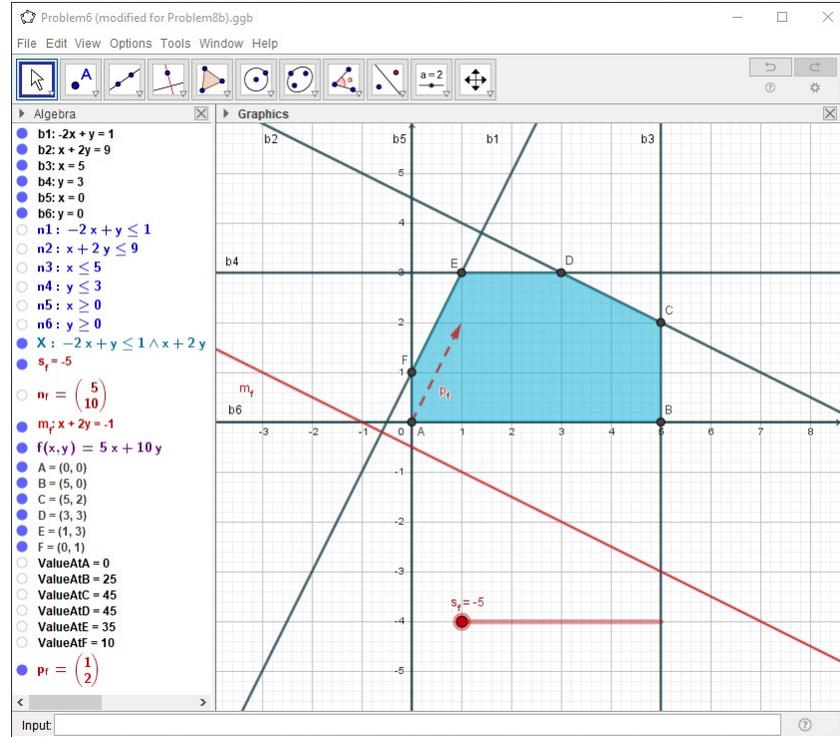


Figure 13: Result after changes for Problem 8.b).

Some additional remarks before we finish this section.

**Remark 9.** In the solution of the part a) in the Problem 8 we briefly described how we can use GeoGebra for calculation of the coordinates of one of the extreme points. Since we had previously entered all the equations of the bounding lines, we may repeat this procedure for every other possible combination of  $b_1, b_2, b_3, b_4, b_5$  and  $b_6$  (in total  $C_6^2 = \frac{6!}{2! \cdot 4!} = 15$ ), and thus finish the first step in the algebraic solution to the LP problem.

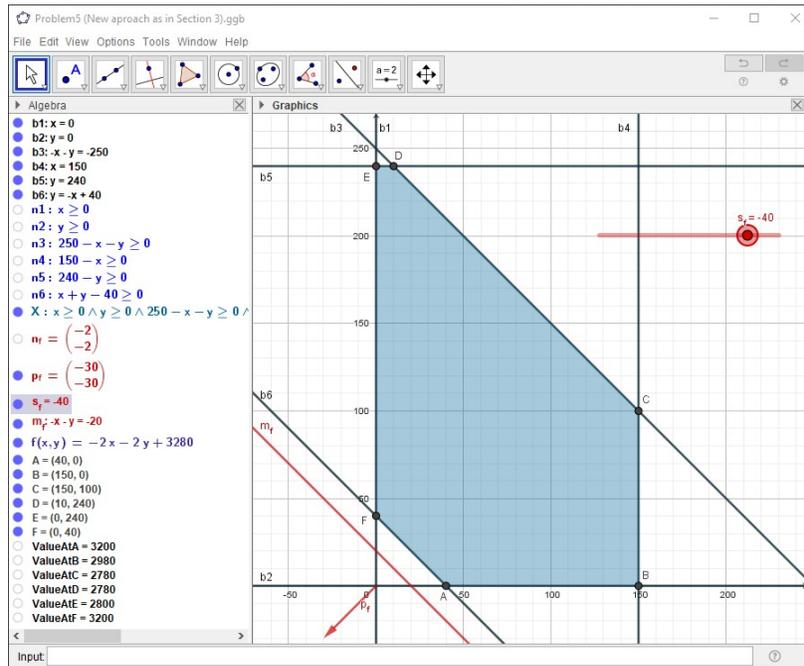
**Remark 10.** In the solution of the part b) in Problem 8, we gave an example of a function that attains its maximum value at infinitely many points of the feasible region. This situation, which is highly probable whenever one or more of the bounding lines is also an isoprofit/isocost/objective-function-line, may occur even in real life problems. Let's look again at the Problem 5 from Section 2. Table 2 in the solution of this problem depends only on the number of units produced at A and B, and the quantities demanded by the stores (i.e., with the unknowns  $x$  and  $y$  it is totally independent from the transportation costs per unit). Let's change one of values in Table 1. We'll increase the transportation

cost per unit from A to II. Instead of 3 we'll take 5. So, we will consider the following table for the transportation costs per unit:

**Table 5:** Modified transportation costs per unit for Problem 5

Store Plant	I	II	III
A	4	5	5
B	5	6	4

In this case, the objective function will be  $L = -2x - 2y + 3280$ . The result obtained with GeoGebra according to the instructions in this section is given in Figure 14.



**Figure 14:** GeoGebra file for modified Problem 5

**Note:** If we create a GeoGebra file as described in this section, then we must add a vector  $p_f := 30 \cdot n_f$  (or some other value for  $q$  above 25 so the this vector is visible in the Graphics view) and set the values for the slider around from about  $-700$  to  $100$  and set the Increment value as  $10$ . Observe the direction of the movement of  $m_f$  (it seems opposite of “from left to right” as we drag the dot on the slider from left to right i.e., when the  $s_f$  increases). The equations in the Algebra window are entered as described in this section, but GeoGebra sometimes automatically overwrites the equalities. For example, the equation  $250 - x - y = 0$  is automatically overwritten as  $-x - y = -250$ .

From Figure 14, we have that the minimum of  $L = -2x - 2y + 3280$  is 2780, and is attained both at point  $C(150,100)$  and point  $D(10,240)$ . This means that the minimum is attained on the segment with end points at  $C$  and  $D$  i.e., at infinitely many points. But this time we have a special case of LP problem, an integer programming problem, which adds another constrain upon the solution. The values of  $x$  and  $y$  must also be integers (we don't transport  $3\frac{1}{2}$ , 2.8 or 0.3 units).

Hence, we don't have infinitely many solutions for the problem. But we've still end up with "too many". How many? Exactly 141 (140 plus Table 3). This means that, in addition to Table 3, we can create another 140 different tables, for which the total cost of transportation will be the same (2780). Enough to fill approximately another 18 pages of this paper. So, we end up here.

#### 4. CONCLUSION

Graphically solving problems in any mathematics topic on a sheet of paper often ends up with an incorrect solution. The result is usually incorrect because students make mistakes when drawing, the drawing can be unprecise (and precision is often very important), manual drawing takes a lot of time, to create an accurate drawing students need to prepare appropriate tools, etc. In this paper we offer a method to increase the interest in mathematics with using learning software which will help for greater curiosity and increased motivation to work and solve problems. For graphical solution of linear programming problems, we've used GeoGebra. With this software students can check if their solution is correct, to get a solution in advance which will guide them to correctly solve the problem on paper, to get solution in short time, etc. Thus, students will be motivated to study mathematics and to achieve better results. This is confirmed by the results of the previously mentioned questionnaire, which was answered by group of 20 students. We encourage, both teachers and students, to use free software, as GeoGebra, while solving the math problems. It is a software that has been used for a long time in education around the world and research shows that its use gives better results.

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