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SEVERAL LOCI GENERATED BY A MOVING TRIANGLE BETWEEN TWO FIXED CIRCLES

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Abstract. According to a famous theorem belonging to the French mathematician Poncelet, if two conics are located in the plane in such a way that a polygon exists which is inscribed in one of the curves and circumscribed with respect to the other one, then each point on any of the two conics generates an inscribed-circumscribed polygon with respect to them. Particularly, two circles could be located in the plane in such a way that one of them is circumscribed with respect to a triangle and the other one is inscribed in it. In connection with this configuration when the triangle moves between the two circles, several loci are considered which are determined by some notable points in the plane of the triangle. It turns out that the loci are circles and ellipses with centers on the central line of the two fixed circles.

Keywords: inscribed circle, circumscribed circle, center of gravity, orthocenter, Euler line, Euler circle, Nagel point, GSP (Geometer's sketchpad)

1. Introduction

It is well-known a remarkable theorem of Poncelet in the Euclidean geometry, a particular case of which is the following assertion:

Theorem. *If the circles Γ and ω are positioned in the plane in such a way that a triangle exists which is inscribed and circumscribed with respect to Γ and ω, respectively, then:*

1) each point on Γ is a vertex of a unique triangle, which is inscribed in Γ and circumscribed with respect to ω;

2) each point on ω is a tangent point on a side of a unique triangle, which is inscribed in Γ and circumscribed with respect to ω.

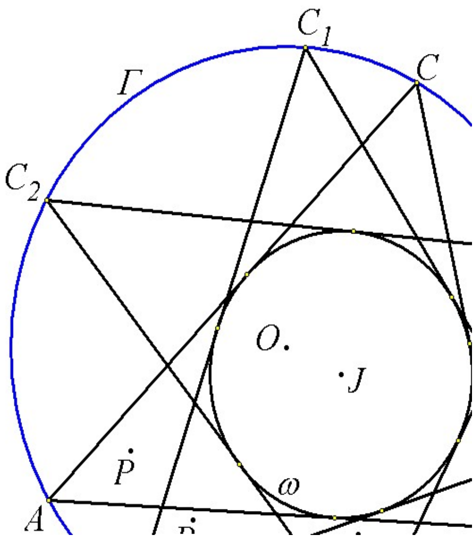


Figure 1

Let Γ and ω be two non-concentric circles satisfying the theorem. It follows from the first item that if A is an arbitrary point on the circle Γ, then there exist such points B and C on the circle in question, that the triangle ABC is circumscribed with respect to ω (Fig. 1). If the point A moves on Γ (the point A occupies consecutive positions A_1, A_2, $\ldots$ on Γ), the triangle ABC will move (ABC occupies consecutive positions $A_1B_1C_1$, $A_2B_2C_2$, $\ldots$) between the circles Γ and ω in such a way that it is inscribed in Γ always and also circumscribed with respect to ω (Fig. 1). Along this motion an arbitrary point P connected with ΔABC in some way will move together with the triangle (P occupies consecutive positions P_1, P_2, $\ldots$ together with the corresponding triangles $A_1B_1C_1$, $A_2B_2C_2$, $\ldots$) (Fig. 1) and at the same time it will describe a determined trajectory in the plane of the circles. Thus, a problem appears to determine the trajectories which some notable points of the triangle describe when the triangle moves between the circles Γ and ω in the mentioned way.

Let O and R be the center and the radius of the circle Γ, respectively, while J and r be the center and the radius of the circle ω, respectively. Loci will be presented that are described by some classic notable points of the triangle. They are noticed by means of the program software Geometer's sketchpad (GSP).

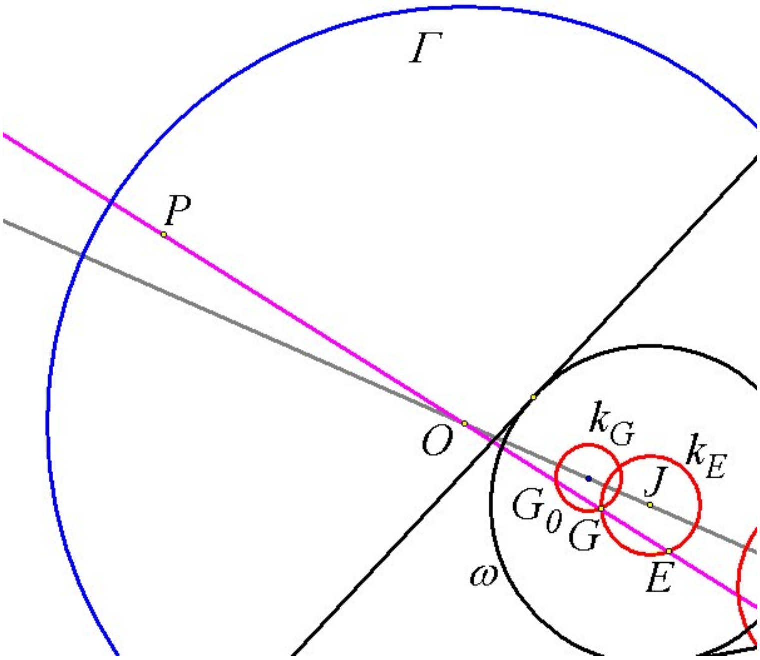


Figure 2

2. Some notable circles in the plane of the triangle

Some notable points of the triangle ABC describe circles along the motion of the triangle between the circles Γ and ω. The following assertions are satisfied for three points which are characteristic for the Euler line:

Theorem 1. *The orthocenter* H *of* ΔABC *describes a circle* k_H *with center* H_0 *on the line* OJ *and radius* $R-2r$ (Fig. 2).

Theorem 2. *The center of gravity* G *of* ΔABC *describes a circle* k_G *with center* G_0 *on the line* OJ *and radius* $\frac{R-2r}{3}$ (Fig. 2).

Theorem 3. *The center* E *of the Euler circle describes a circle* k_E *with center* J *and radius* $\frac{R-2r}{2}$ (Fig. 2).

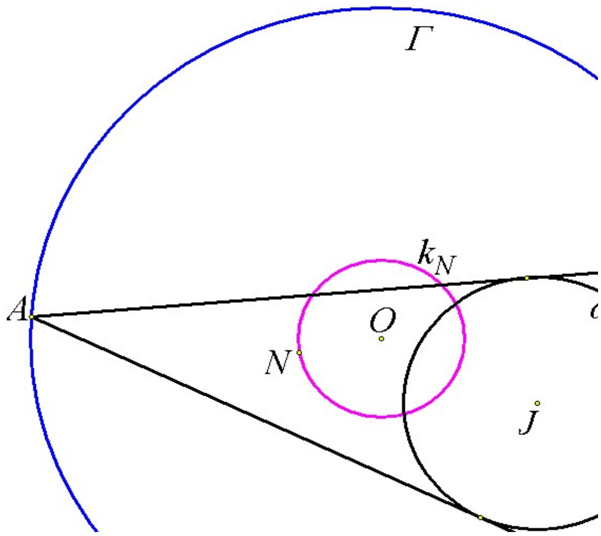
Proofs and generalizations of these assertions are included in [2].

Other notable points of ΔABC are the points of Nagel and Gergonne. If r_a, r_b and r_c are the radii of the ex-circles of ΔABC, which are tangent to the sides BC, CA and AB, respectively, then the Nagel point N and the Gergonne one G could be determined with the following vector equalities, respectively:

$$\overrightarrow{ON} = \frac{r_b r_c \overrightarrow{OA} + r_c r_a \overrightarrow{OB} + r_a r_b \overrightarrow{OC}}{r_b r_c + r_c r_a + r_a r_b}, \quad \overrightarrow{OG} = \frac{r_a \overrightarrow{OA} + r_b \overrightarrow{OB} + r_c \overrightarrow{OC}}{r_a + r_b + r_c}.$$

The points in question satisfy the following two assertions:

Theorem 4. *The Nagel point* N *describes a circle* k_N *with center* O *and radius* $R-2r$ (Fig. 3).



Figure

A proof and a generalization of this assertion are included in [2].

Theorem 5. *If G is the Gergonne point of a moving triangle ABC between the circles Γ and ω, then it describes a circle $k(G)$ with center P on the line OJ, where $OP = \frac{4(R+r).OJ}{4R+r}$, and a radius $\rho = \frac{(R-2r)r}{4R+r}$* (Fig. 4).

Two different proofs of this assertion are included in [3] and [4]. The circle $k(G)$ will be called Poncelet-Gergonne circle.

The points on the Euler circle of ΔABC describe a special set of circles. More precisely, it is satisfied the following:

Theorem 6. *If M is a point on the Euler circle of the triangle ABC, moving between the circles Γ and ω, then it describes a circle $k(M)$ with radius $\rho = \frac{1}{2}(R-2r)$, which is tangent to ω exteriorly* (Fig. 5).

The proof of this theorem will be published in Mathematics Plus journal later. The circles $k(M)$ will be called Poncelet-Euler circles.

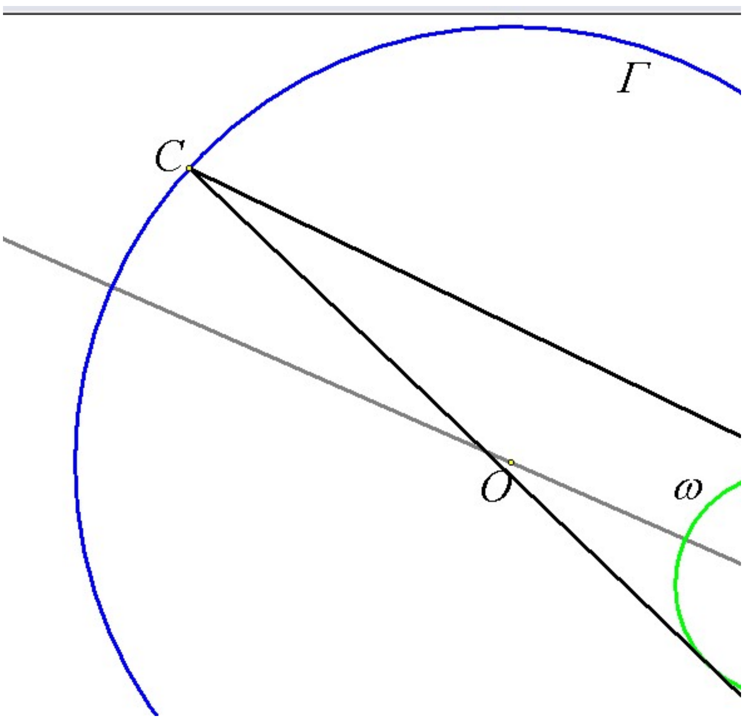


Figure 4

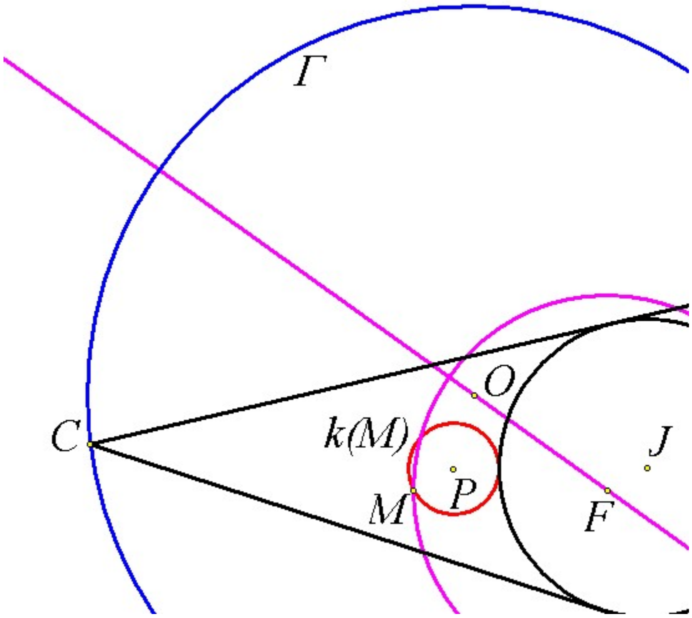


Figure 5

3. Two notable ellipses in the plane of the triangle

The points L, L_a, L_b and L_c, determined with the vector equalities

$$\overrightarrow{OL}=\frac{a_0^2\overrightarrow{OA}+b_0^2\overrightarrow{OB}+c_0^2\overrightarrow{OC}}{a_0^2+b_0^2+c_0^2},\ \overrightarrow{OL_a}=\frac{-a_0^2\overrightarrow{OA}+b_0^2\overrightarrow{OB}+c_0^2\overrightarrow{OC}}{-a_0^2+b_0^2+c_0^2},$$

$$\overrightarrow{OL_b}=\frac{a_0^2\overrightarrow{OA}-b_0^2\overrightarrow{OB}+c_0^2\overrightarrow{OC}}{a_0^2-b_0^2+c_0^2},\ \overrightarrow{OL_c}=\frac{a_0^2\overrightarrow{OA}+b_0^2\overrightarrow{OB}-c_0^2\overrightarrow{OC}}{a_0^2+b_0^2-c_0^2},$$

where $BC=a_0$, $CA=b_0$ and $AB=c_0$, are called Lemoine points of ΔABC.

It turns out that the Lemoine points of ΔABC describe ellipses with interesting properties. The characteristic properties of these ellipses are described in the following assertions.

Theorem 7. *If L is the Lemoine point of the triangle ABC, moving between the circles Γ and ω, then it describes an ellipse $k(L)$ with center T on the line OJ, where $OT=\dfrac{3R^2}{3R^2-2Rr+r^2}\cdot OJ$. The value of the small semiaxis α of*

$k(L)$ on OJ is equal to $\alpha = \dfrac{Rr(R-2r)}{3R^2-2Rr+r^2}$, while the value of the big one is equal to $\beta = R(R-2r)\sqrt{\dfrac{r}{(4R+r)(3R^2-2Rr+r^2)}}$ (Fig. 6).

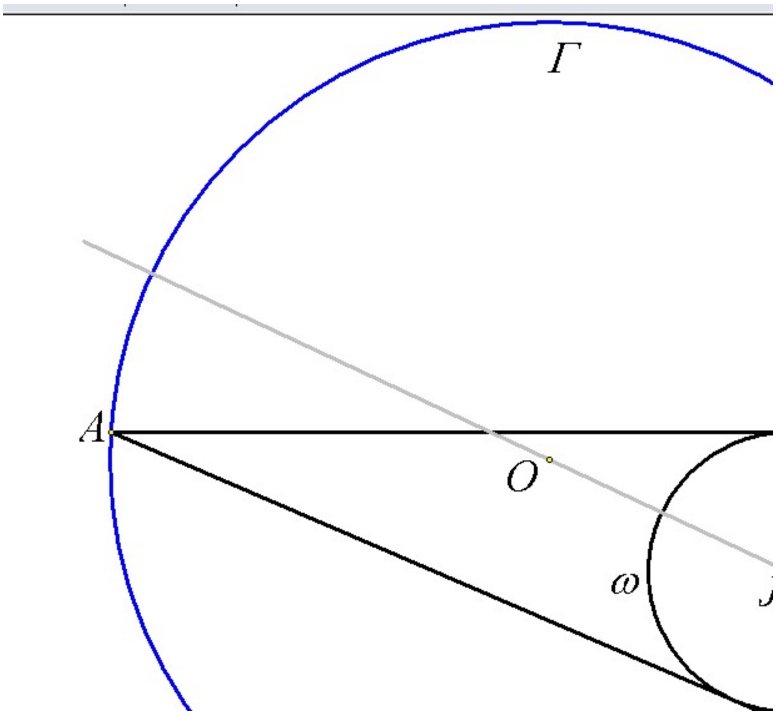


Figure 6

The ellipse will be called Poncelet-Lemoine ellipse.

Theorem 8. *If L_a, L_b and L_c are the outer Lemoine points of the triangle ABC moving between the circles $\Gamma(O,R)$ and $\omega(J,r)$, then they describe a curve $\bar{k}(L)$ of second degree with a focus O and a focal axis OJ. The curve $\bar{k}(L)$ has the following properties:*

1) If $2r \le R < (\sqrt{2}+1)r$, then the curve $\bar{k}(L)$ is an ellipse with a small semiaxis $\alpha = \dfrac{R^2 r}{r^2+2Rr-R^2}$, big semiaxis $\beta = \dfrac{R^2}{\sqrt{r^2+2Rr-R^2}}$ and a center T, satisfying the equality $OT = -\dfrac{R^2.OJ}{r^2+2Rr-R^2}$;

2) If $R > \left(\sqrt{2}+1\right)r$*, then the curve* $\bar{k}(L)$ *is a hyperbola with a small semiaxis* $\alpha = \frac{R^2 r}{R^2 - 2Rr - r^2}$*, big semiaxis* $\beta = \frac{R^2}{\sqrt{R^2 - 2Rr - r^2}}$ *and a center* T*, satisfying the equality* $OT = \frac{R^2.OJ}{R^2 - 2Rr - r^2}$*;*

3) If $R = \left(\sqrt{2}+1\right)r$*, then the curve* $\bar{k}(L)$ *is a parabola with a focal parameter* $p = \left(\sqrt{2}+1\right)^2 r$ *and a vertex* V*, satisfying the equality* $OV = \frac{\left(\sqrt{2}+1\right)^2 r}{2}$.

The ellipse $\bar{k}(L)$ will be called outer Poncelet-Lemoine ellipse.

Detailed proofs of these theorems will be published in the Mathematics Plus journal.

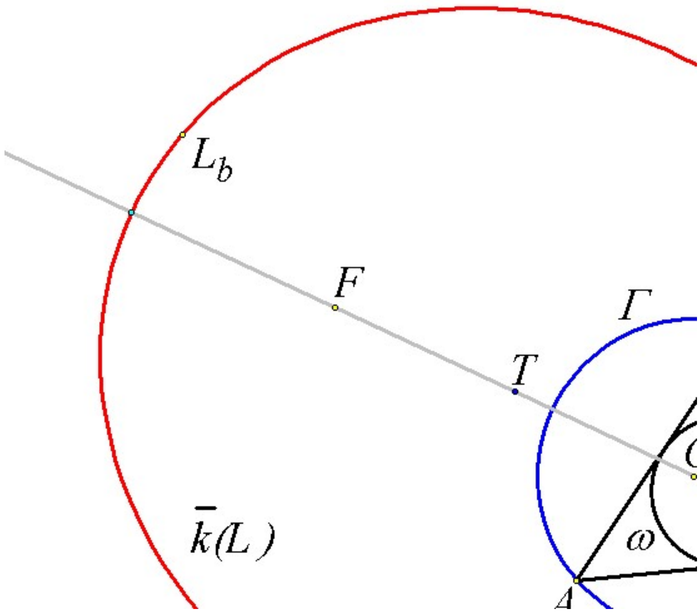


Figure 7

Concluding we could state that the described circles and ellipses in the listed theorems are noticed by means of the Poncelet theorem and the geometric capabilities of the program software GSP. They are located in the plane of a scalene triangle moving between two fixed circles. The curves themselves exist for any triangle, no matter it is regarded as moving or stationary one. For this reason they could be called notable circles and ellipses of the triangle.

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