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GENERALIZATION OF A REICH-TYPE CONTRACTIVE MAPPING IN A COMPLETE METRIC SPACE

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Abstract. In this work, is given a generalization of the fixed point theorem of the Reich-type mapping on a complete metric space (X,d). Continuous, injective and sequentially convergent mapping T was used, as well as function f is from the class Θ continuous monotonically nondecreasing functions $f:[0,+\infty)\to[0,+\infty)$ such that $f^{-1}(0)=\{0\}$, where it is additionally taken the function to be subadditive, i.e. $f(x+y)\le f(x)+f(y)$, for each $x,y\in[0,+\infty)$.

1. Introduction

Banach's principle for a fixed point is well known in the literature, namely:

Let (X,d) is a metric space. Mapping $S:X\to X$ we will call it a contraction if there exits $\lambda\in(0,1)$ such that for each $x,y\in X$ is true

$$d(Sx,Sy)\le\lambda d(x,y). \tag{1}$$

If metric space (X,d) is complete, then the mapping T for which condition (1) is satisfied has a unique fixed point. In 1968, R. Kannan ([4]) generalized Banach's fixed point principle as follows:

Theorem 1. If mapping $S:X\to X$ where (X,d) is complete metric space, satisfies the inequality

$$d(Sx,Sy)\le\lambda(d(x,Sx)+d(y,Sy)), \tag{2}$$

where $\lambda\in(0,\frac{1}{2})$ and $x,y\in X$, then S has a single fixed point. ■

If S satisfies the condition (2), then for S we say it is a Kannan-type mapping.

In 1972, similar contraction conditions were introduced by S. K. Chatterjea ([7]), as follows:

Theorem 2. If mapping $S:X\to X$ where (X,d) is a complete metric space satisfies the inequality

$$d(Sx,Sy)\le\lambda(d(x,Sy)+d(y,Sx)), \tag{2}$$

where $\lambda\in(0,\frac{1}{2})$ and $x,y\in X$, then S has a single fixed point. ■

If S satisfies condition (2), then we say that is a Chatterjea-type mapping.

In 1971, S. Reich ([3]), gave a new generalization of Banach's fixed point principle as follows:

Theorem 3. If mapping $S: X \to X$ where (X,d) is a complete metric space satisfies the inequality

$$d(Sx,Sy) \le ad(x,Sx) + bd(y,Sy) + cd(x,y), \tag{3}$$

where $a > 0, b > 0$ and $c > 0$ are such that $a+b+c<1$ and $x, y \in X$, then S има has a single fixed point. ■

If it satisfies condition (3), then we say that is a Reich-type mapping.

In [9] S. Moradi and D. Alimohammadi generalize R. Kannan's result, using the sequentially convergent mappings, and in [1] several generalizations of Kannan and Chatterjea's theorems are proved, using the sequentially convergent mappings and , which are defined as follows:

Definition 1 ([8]). Let (X,d) be a metric space. A mapping $T: X \to X$ is said sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ also is convergence.

In [8] S. Moradi and A. Beiranvand introduce the concept of T_f contractive mapping, whereby they use the class Θ of continuous monotonically nondecreasing functions $f:[0,+\infty) \to [0,+\infty)$ such that $f^{-1}(0) = \{0\}$, which is defined as follows.

Definition 2 ([8]). Let (X,d) be a metric space, $S, T: X \to X$ and $f \in \Theta$. A mapping S is said T_f −contraction if there exist $\lambda \in (0,1)$ such that

$$f(d(TSx,TSy)) \le \lambda f(d(Tx,Ty)),$$

for all $x, y \in X$.

Let us note here that, if $f \in \Theta$, then from $f^{-1}(0) = \{0\}$ follows that $f(t) > 0$, for each $t > 0$. S. Moradi and A. Beiranvand prove that if S is T_f contractive mapping, then S has a single fixed point. Then, in [2] M. Kir and H. Kiziltunc generalize the result of S. Moradi and A. Beiranvand for the mappings of the type of Kannan and Chatterjea. In [10] are generalized the results of Kir and Kiziltunc and is given their application.

In the following considerations we will give an analogous generalization for the Reich-type mapping.

2. MAINS RESULTS

Theorem 4. Let (X,d) is a complete metric space $S: X \to X$, $f \in \Theta$ is such that $f(p+q) \le f(p) + f(q)$, for each $p, q \in [0,+\infty)$ and mapping $T: X \to X$ is continuous, injection and sequentially convergent. If exits $a > 0, b > 0$ and $c > 0$ such that $a+b+c<1$ and

$$f(d(TSx,TSy)) \le af(d(Tx,TSx)) + bf(d(Ty,TSy)) + cf(d(Tx,Ty)) \tag{4}$$

for each $x, y \in X$, then S has a single fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to fixed point.

Proof. Let x_0 is an arbitrary point from X and let the array $\{x_n\}$ is determined by $x_{n+1} = Sx_n$, $n = 0,1,2,3,...$. It follows from the inequality (4).

$$\begin{aligned} f(d(Tx_{n+1},Tx_n)) &= f(d(TSx_n,TSx_{n-1})) \\ &\le af(d(Tx_n,TSx_n)) + bf(d(Tx_{n-1},TSx_{n-1})) + cf(d(Tx_n,Tx_{n-1})) \\ &= af(d(Tx_n,Tx_{n+1})) + (b+c)f(d(Tx_{n-1},Tx_n)), \end{aligned}$$

i.e.

$$f(d(Tx_{n+1},Tx_n)) \le \frac{b+c}{1-a} f(d(Tx_n,Tx_{n-1})) .$$

Therefore, for $\lambda = \frac{b+c}{1-a} < 1$ the following holds true

$$f(d(Tx_{n+1},Tx_n)) \le \lambda f(d(Tx_n,Tx_{n-1})) , \tag{5}$$

for each $n = 1,2,3,...$. From the inequality (5) it follows that

$$f(d(Tx_{n+1},Tx_n)\,) \le \lambda^n f(d(Tx_1,Tx_0)) , \tag{6}$$

for each $n = 1,2,3,...$. Now from inequality (6) the properties of the metric and the monotonicity and subadditivity of the function f it follows that for each $m,n \in R$ $n > m$ the following holds true

$$\begin{aligned} f(d(Tx_n,Tx_m)) &\le f(\sum_{k=m}^{n-1} d(Tx_{k+1},Tx_k)) \le \sum_{k=m}^{n-1} f(d(Tx_{k+1},Tx_k)) \\ &\le \sum_{k=m}^{n-1} \lambda^k f(d(Tx_1,Tx_0)) < \frac{\lambda^m}{1-\lambda} f(d(Tx_1,Tx_0)). \end{aligned}$$

It follows from the last inequality

$$\lim_{m,n\to\infty} f(d(Tx_n,Tx_m)) = 0 ,$$

and because $f \in \Theta$ we have $\lim_{m,n\to\infty} d(Tx_n,Tx_m) = 0$. According to that, $\{Tx_n\}$ is Cauchy sequence. But, X is a complete metric space, so the sequence $\{Tx_n\}$ is convergent. Further, the mapping $T: X \to X$ e sequentially convergent, so therefore the sequence $\{x_n\}$ is convergent i.e. exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. From the continuity of T it follows $\lim_{n\to\infty} Tx_n = Tu$.

We will prove that $u \in X$ is a fixed point for the mapping S. We have

$$\begin{aligned} f(d(TSu,Tx_{n+1})) &= f(d(TSu,TSx_n)) \\ &\le af(d(TSu,Tu)) + bf(d(TSx_n,Tx_n)) + cf(d(Tu,Tx_n)) \\ &= af(d(TSu,Tu)) + bf(d(Tx_{n+1},Tx_n)) + cf(d(Tu,Tx_n)). \end{aligned}$$

If in the last inequality we take $n \to \infty$, then form $\lim\limits_{n\to\infty} Tx_n = Tu$ and the continuity of metric and function f follows the inequality

$$f(d(TSu,Tu)) \le \tfrac{b+c}{1-a} f(0).$$

But, $0 < \tfrac{b+c}{1-a} < 1$ and $f^{-1}(0) = \{0\}$, so it follows from inequality $d(TSu,Tu) = 0$, i.e. $TSu = Tu$. Finally, T is injection and therefore $Su = u$, i.e. mapping S has a fixed point.

Let $u, v \in X$ are two fixed points for S, i.e. $Su = u$ and $Sv = v$. From the inequality, (4) it follows that

$$f(d(Tu,Tv)) = f(d(TSu,TSv)) \le af(d(Tu,TSu)) + bf(d(Tv,TSv)) + cf(d(Tu,Tv))$$

i.e.

$$f(d(Tu,Tv)) \le \tfrac{a+b}{1-c} f(0),$$

so similarly as above we conclude that $d(Tu,Tv) = 0$. Therefore, $Tu = Tv$. But, T is an injection, and therefore $u = v$, i.e. S has a single fixed point.

Finally, from the arbitrariness of the point x_0 it follows that for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to the fixed point. ■

Corollary 1. Let (X,d) is a complete metric space, $S : X \to X$ and $f \in \Theta$ is such that $f(p+q) \le f(p) + f(q)$, for each $p,q \in [0,+\infty)$. If $a > 0, b > 0, c \ge 0$ exits such that $a + b + c < 1$ and

$$f(d(Sx,Sy)) \le af(d(x,Sx)) + bf(d(y,Sy)) + cf(d(x,y)),$$

for each $x, y \in X$, then S has a single fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to the fixed point.

Proof. The mapping $Tx = x$, for each $x \in X$ is continuous, injection and sequentially convergent. Therefore the corollary follows directly from Theorem 4 for $Tx = x$. ■

Corollary 2. Let (X,d) is a complete metric space, $S : X \to X$ and mapping $T : X \to X$ is continuous and sequentially convergent. If $a > 0, b > 0, c \ge 0$ exits such that $a + b + c < 1$ and

$$d(TSx,TSy) \le ad(Tx,TSx) + bd(Ty,TSy) + cd(Tx,Ty)$$

for each $x, y \in X$, then S has a single fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to the fixed point.

Proof. The function $f(t) = t$, $t \ge 0$ is monotonically nondecreasing, $f^{-1}(0) = \{0\}$ and is such that $f(p+q) \le f(p) + f(q)$, for each $p,q \in [0,+\infty)$. Therefore the corollary follows directly from Theorem 4 for $f(t) = t$. ■

Comment. If we consider that the mapping $Tx = x$, for each $x \in X$ is continuous, injection and sequentially convergent, from corollary 2 follows theorem 3, [3], i.e. follows that if for the mapping $S:X\to X$ exits $a>0, b>0, c\geq 0$ such that $a+b+c<1$ and

$$d(Sx,Sy)\leq ad(x,Sx)+bd(y,Sy)+cd(x,y)$$

for each $x,y\in X$, then S has a single fixed point.

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