ISBN 978-608-4904-04-5(електронско издание)
ISBN 978-608-4904-03-8(печатено издание)
UDC: 515.124:515.126.4

COMMON FIXED POINTS OF TWO T_f REICH-TYPE CONTRACTIONS IN COMPLETE METRIC SPACE

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Abstract. In this work, it considered theorems about common fixed points of two T_f Reich-type contractions in complete metric space (X,d). In doing so, it is used that the mapping T is continuous, injection and sequentially convergent, and function f is from the class Θ continuous monotonically nondecreasing functions $f:[0,+\infty)\to[0,+\infty)$ such that $f^{-1}(0)=\{0\}$, where it is additionally taken the function to be subadditive, i.e. $f(p+q)\le f(p)+f(q)$, for each $p,q\in[0,+\infty)$.

1. INTRODUCTION

In literature there are well known Banach's fixed point principle and its generalizations given by R. Kannan ([4]), S. K. Chatterjea ([7]), P.V. Koparde, B. B. Waghmode ([3]) and Reich ([10]).

In [9] S. Moradi and D. Alimohammadi generalize the result of R. Kannan, using the sequentially convergent mappings.

Then, in [1] several generalizations of the theorems of R. Kannan, S. K. Chatterjea and P. V. Koparde, B. B. Waghmode were proved, using the sequentially convergent mappings, and in 2016 in [5] with the help of sequentially convergent mappings are proven more common fixed point results for two type mappings by R. Kannan, S. K. Chatterjea and P.V. Koparde, B.B. Waghmode.

In 2010 in [8] S. Moradi and A. Beiranvand introduce the concept of T_f contractive mapping, using the Θ class of continuous monotonically nondecreasing functions $f:[0,+\infty)\to[0,+\infty)$ such that $f^{-1}(0)=\{0\}$. Note here that, if $f\in\Theta$, then from $f^{-1}(0)=\{0\}$ follows that $f(t)>0$, for each $t>0$.

S. Moradi and A. Beiranvand prove that if S is T_f contractive mapping, and then S has a unique fixed point.

Then, in 2014 M. Kir and H. Kiziltunc generalize the result of S. Moradi and A. Beiranvand to mappings of type R. Kannan and S. K. Chatterjea.

In 2021 in [11], [12] and [13] are proven more generalizations for common fixed points of two T_f contractions of the type of R. Kannan, S. K. Chatterjea and P. V. Koparde, B. B. Waghmode on a complete metric space, while for the

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2010 *Mathematics Subject Classification.* Primary: Functional Analysis. *Key words and phrases*. Keywords should be put here

function f from the class Θ it is further assumed that it is subadditive, i.e. that $f(p+q) \le f(p)+f(q)$, for each $p,q \in [0,+\infty)$. In further considerations under the same assumptions we will give some results for common fixed points of two contractions of the Reich type.

2. MAIN RESULT

Definition 1 ([8]). Let (X,d) be a metric space. A mapping $T: X \to X$ is sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ also is convergence.

Definition 2 ([8]). Let (X,d) be a metric space, $S,T: X \to X$ and $f \in \Theta$. A mapping S is T_f – contraction if there exist $\lambda \in (0,1)$ such that

$$f(d(TSx,TSy)) \le \lambda f(d(Tx,Ty)),$$

for all $x,y \in X$.

Theorem 1. Let (X,d) is a complete metric space, $S_1,S_2 : X \to X$, $f \in \Theta$ is such that $f(p+q) \le f(p)+f(q)$, for each $p,q \in [0,+\infty)$ and mapping $T: X \to X$ is continuous, injection and sequentially convergent. If there are any $a,b>0$ and $c \ge 0$ such that $a+b+c \in (0,1)$ and

$$f(d(TS_1x,TS_2y)) \le af(d(Tx,TS_1x))+bf(d(Ty,TS_2y))+cf(d(Tx,Ty)) \tag{1}$$

for each $x,y \in X$, then S_1 and S_2 have a single common fixed point.

Proof. Let x_0 is an arbitrary point from X and let the sequence $\{x_n\}$ is defined by

$$x_{2n+1} = S_1x_{2n},\ x_{2n+2} = S_2x_{2n+1},\ n = 0,1,2,3,\dots .$$

If it exists $n \ge 0$, such that $x_n = x_{n+1} = x_{n+2}$, then it is easily proved that $u = x_n$ is a common fixed point of S_1 and S_2. Let us therefore assume that there are no three consecutive equal members of the sequence $\{x_n\}$. Then, using inequality (1), it is easy to prove that the following inequalities are true:

$$f(d(Tx_{2n+1},Tx_{2n})) \le \tfrac{b+c}{1-a} f(d(Tx_{2n},Tx_{2n-1}))$$

and

$$f(d(Tx_{2n},Tx_{2n-1})) \le \tfrac{a+c}{1-b} f(d(Tx_{2n-1},Tx_{2n-2})).$$

From the last two inequalities it follows that for each $n = 0,1,2,\dots$ and for

$$\lambda = \min\{\tfrac{a+c}{1-b},\tfrac{b+c}{1-a}\} \in (0,1)$$

is true:

$$f(d(Tx_{n+1},Tx_n)) \le \lambda f(d(Tx_n,Tx_{n-1})) . \qquad (2)$$

Furthermore, from inequality (2) it follows

$$f(d(Tx_{n+1},Tx_n)) \le \lambda^n f(d(Tx_1,Tx_0)) , \qquad (3)$$

for each $n = 0,1,2,...$. Now from the metric properties, the function properties f and the inequality (3) follows that for each $m,n \in R$, $n > m$ is true

$$\begin{aligned} f(d(Tx_n,Tx_m)) &\le f(\sum_{k=m}^{n-1} d(Tx_{k+1},Tx_k)) \le \sum_{k=m}^{n-1} f(d(Tx_{k+1},Tx_k)) \\ &\le \sum_{k=m}^{n-1} \lambda^k f(d(Tx_1,Tx_0)) < \tfrac{\lambda^m}{1-\lambda} f(d(Tx_1,Tx_0)). \end{aligned}$$

It follows from the last inequality that

$$\lim_{m,n\to\infty} f(d(Tx_n,Tx_m)) = 0 ,$$

and because $f \in \Theta$ we have $\lim_{m,n\to\infty} d(Tx_n,Tx_m) = 0$. Therefore, the sequence $\{Tx_n\}$ is Cauchy and because (X,d) is a complete metric space it is convergent. Further, the mapping $T: X \to X$ e sequentially convergent, so therefore the sequence $\{x_n\}$ is convergent, i.e. exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Now, from the continuity of T it follows $\lim_{k\to\infty} Tx_n = Tu$.

We will prove that $u \in X$ is a fixed point for the mapping S_1 . We have:

$$\begin{aligned} f(d(Tu,TS_1u)) &\le f(d(Tu,Tx_{2n+2})) + f(d(Tx_{2n+2},TS_1u)) \\ &= f(d(Tu,Tx_{2n+2})) + f(d(TS_2x_{2n+1},TS_1u)) \\ &\le f(d(Tu,Tx_{2n+2})) + af(d(Tu,TS_1u)) + bf(d(Tx_{2n+1},TS_2x_{2n+1})) + cf(d(Tu,Tx_{2n+1})) \\ &= f(d(Tu,Tx_{2n+2})) + af(d(Tu,TS_1u)) + bf(d(Tx_{2n+1},Tx_{2n+2})) + cf(d(Tu,Tx_{2n+1})) \end{aligned}$$

The mappings f and T are continuous, therefore, from the properties of the metric, works follows if in the last inequality we take $n \to \infty$, we will have

$$(1-a) f(d(Tu,TS_1u)) \le (1+b+c) f(0)$$

But, $1-a > 0$ and $f^{-1}(0) = \{0\}$, so from so from the last inequality we have $d(Tu,TS_1u) = 0$, i.e. $TS_1u = Tu$. Finally, T is an injection, therefore $S_1u = u$, which means that u is a fixed point for the mapping S_1 . Analogously it is proved that u is a fixed point for the mapping S_2 , i.e. u is a common fixed point for the mappings S_1 and S_2 .

We will prove that S_1 and S_2 have a single common fixed point. Let $v \in X$ is a fixed point for S_2 , i.e. $S_2v = v$. Then

$$\begin{aligned} f(d(Tu,Tv)) &= f(d(TS_1u,TS_2v)|) \le af(d(Tu,TS_1u)) + bf(d(Tv,TS_2v))) + cf(d(Tu,Tv) \\ &= af(d(Tu,Tu)) + bf(d(Tv,Tv))) + cf(d(Tu,Tv) \\ &= (a+b)f(0) + cf(d(Tu,Tv). \end{aligned}$$

Now, $1-c>0$ and $f^{-1}(0)=\{0\}$, so from the last inequality we have $d(Tu,Tv)=0$, i.e. holds that $Tu=Tv$. But, T is an injection, so $u=v$, which means that S_1 and S_2 have a single common fixed point. ■

Corollary 1. Let (X,d) is a complete metric space, $S_1,S_2 : X \to X$, $f \in \Theta$ is such that $f(p+q) \le f(p)+f(q)$, for each $p,q \in [0,+\infty)$ and mapping $T : X \to X$ is continuous, injection and sequentially convergent. If $\lambda \in (0,1)$ exists such that

$$f(d(TS_1x,TS_2y)) \le \lambda\sqrt[3]{f(d(Tx,TS_1x)) \cdot f(d(Ty,TS_2y)) \cdot f(d(Tx,Ty)}$$

for each $x,y \in X$, then S_1 and S_2 have a single common fixed point.

Proof. It follows from the inequality between the arithmetic mean and the geometric mean and Theorem 1 for $a=b=c=\frac{\lambda}{3}$.■

Corollary 2. Let (X,d) is a complete metric space, $S_1,S_2 : X \to X$, $f \in \Theta$ is such that $f(p+q) \le f(p)+f(q)$, for each $p,q \in [0,+\infty)$ and mapping $T : X \to X$ is continuous, injection and sequentially convergent. If $a,b>0$ and $c \ge 0$ exits such that $a+b+c \in (0,1)$ and

$$f(d(TS_1x,TS_2y)) \le \frac{af^2(d(Tx,TS_1x))+bf^2(d(Ty,TS_2y))}{f(d(Tx,TS_1x))+f(d(Ty,TS_2y))} + cf(d(Tx,Ty),$$

for each $x,y \in X$, then S_1 and S_2 have a single common fixed point.

Proof. It folloes from the inequality given in the condition, the inequality (1). ■

Corollary 3. Let (X,d) is a complete metric space, $S_1,S_2 : X \to X$, $f \in \Theta$ is such that $f(p+q) \le f(p)+f(q)$, for each $p,q \in [0,+\infty)$ and mapping $T : X \to X$ is continuous, injection and sequentially convergent. If $a,b>0$ exits such that $a+b \in (0,1)$ and

$$f(d(TS_1x,TS_2y)) \le af(d(Tx,TS_1x)) + bf(d(Ty,TS_2y)),$$

for each $x,y \in X$, then S_1 and S_2 have a single common fixed point.

Proof. The corollary follows from Theorem 1, for $c=0$. ■

Corollary 4. Let (X,d) is a complete metric space, $S_1,S_2 : X \to X$ and $f \in \Theta$ is such that $f(p+q) \le f(p)+f(q)$, for each $p,q \in [0,+\infty)$. If $a,b>0$ exits and $c \ge 0$ such that $a+b+c \in (0,1)$ and

$$f(d(S_1x,S_2y)) \le af(d(x,S_1x)) + bf(d(y,S_2y)) + cf(d(x,y))$$

for each $x, y \in X$, then S_1 and S_2 have a single common fixed point.

Proof. Mapping $T : X \to X$ defined with $Tx = x$ is an uninterrupted injection and is sequentially convergent. So, the corollary follows from Theorem 1 for $Tx = x$. ■

Corollary 5. Let (X, d) is a complete metric space, $S_1, S_2 : X \to X$ and $f \in \Theta$ is such that $f(p+q) \le f(p) + f(q)$, for each $p, q \in [0, +\infty)$. If $a, b > 0$ exits such that $a + b \in (0,1)$ and

$$f(d(S_1x, S_2y)) \le af(d(x, S_1x)) + bf(d(y, S_2y)),$$

for each $x, y \in X$, then S_1 and S_2 have a single common fixed point.

Proof. The corollary follows from corollary 3 for $Tx = x$ or from corollary 4 for $c = 0$. ■

Corollary 6. Let (X, d) is a complete metric space, $S_1, S_2 : X \to X$, $f \in \Theta$ is such that $f(p+q) \le f(p) + f(q)$, for each $p, q \in [0, +\infty)$ and mapping $T : X \to X$ is continuous, injection and sequentially convergent. If $p, q \in N$ exits and $a, b > 0$ and $c \ge 0$ such that $a + b + c \in (0,1)$ and

$$f(d(TS_1^p x, TS_2^q y)) \le af(d(Tx, TS_1^p x)) + bf(d(Ty, TS_2^q y)) + cf(d(Tx, Ty))$$

for each $x, y \in X$, then S_1 and S_2 have a single common fixed point. ■

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