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# A PARTICULAR SOLUTION TO THE SPECIAL CASE OF A FOURTH-ORDER SHORTENED LORENZ SYSTEM

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**Abstract.** In this paper, from the expanded class of the second-order linear differential equations, a subclass of the second-order linear differential equations will be obtained. For this subclass, a new condition for reductability according to Frobenius, as well as explicit formulas of its particular solution will be received. This subclass of the second-order linear differential equations and its particular solution, for obtaining a particular solution of the special case of the fourth-order shortened Lorenz system which was obtained from the Modified Lorenz system will be applied.

## 1. INTRODUCTION

For the class of second-order linear differential equations of B.S. Popov in [1] necessary and sufficient condition for reductable according to Frobenius is obtained. In mathematical literature [2-7] the following theorem is known:

**Theorem 1.** Let the differential equation

$$(Ax^2+Bx+C)y''+(Dx+E)y'+Fy=0,\quad A,B,C,D,E,F\in\mathbb{R} \qquad (1)$$

is given. The differential equation (1) is integrable if there exists an integer $n\in\mathbb{Z}$ (the smallest number after absolute value if there are such numbers) that satisfies the condition

$$n(n-1)A+nD+F=0 \qquad (2)$$

In doing so, the differential equation (1) has a particular solution which is given by the formula

$$y_p(x)=P_n(x)=(Ax^2+Bx+C)e^{-\int\frac{Dx+E}{Ax^2+Bx+C}dx}[(Ax^2+Bx+C)^{n-1}e^{\int\frac{Dx+E}{Ax^2+Bx+C}dx}]^{(n)} \qquad (3)$$

if $n\in\mathbb{N}$ (a polynomial solution).

But, if $n\in\mathbb{Z}^-$, $k=-(n+1)\in\mathbb{N}$ then a particular solution will be given by the formula

$$y_p(x)=[(Ax^2+Bx+C)^{k+1}e^{-\int\frac{Dx+E}{Ax^2+Bx+C}dx}]^{(k)} \qquad (4)$$

The Lorenz system in mathematical literature (e.g. [8-20]) is already known. Its explicit solutions are unknown and its behavior is analyzed through graphical visualization (e.g. [8-14]). It has the following form

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$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(r - z) - y$$
$$\dot{z} = xy - bz$$
$$\sigma, r, b > 0$$

and initial values $x_0 = x(0), y_0 = y(0), z_0 = z(0)$ . The Modified Lorenz system in [21] with initial values $x_0 = x(0), y_0 = y(0), z_0 = z(0)$ , $z_p = \overset{(p)}{z}(0)$ , $p \in \{1,2,3,4\}$

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(r - z) - y \qquad (5)$$
$$\overset{(5)}{z} = -(A + b)\overset{(4)}{z} + (B - Ab)\dddot{z} - (C - Bb)\ddot{z} + (D - Cb)\dot{z} + Dbz$$
$$\sigma, r, b > 0,\ A = 1 + \sigma + b, B = \sigma(r - z_0) - x_0^{\ 2}, C = \sigma\, x_0 y_0, D = -\sigma^2 y_0^{\ 2}$$

is presented.

The third equation of the Modified Lorenz system is a five-order linear homogeneous differential equation with the constant coefficients. Its characteristic equation is

$$m^5 + (A + b)m^4 - (B - Ab)m^3 + (C - Bb)m^2 - (D - Cb)m - Db = 0 \qquad (6)$$

which solutions are $m_1 = -b$, $m_{2/3/4/5} = k(A,B,C,D,b)$. The explicit solutions of the Modified Lorenz system in [21] for any value of the parameters $\sigma, r, b > 0$ σ>0 and initial values $x_0, y_0, z_0$ are obtained.

By using of two solutions from the solutions $m_{1/2/3/4/5}$ of the equation (5), the 7th order Modified Lorenz system (5) in [22] is transformed in a fourth-order subsystem Modified Lorenz system

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(r^* - z) - y \qquad (6)$$
$$\dot{z} = u$$
$$\dot{u} = (m_1 + m_2)u - m_1 m_2 z$$
$$\sigma, r^* > 0,\ m_1, m_2 \in \mathbb{R}$$

with the initial values $x_0 = x(0), y_0 = y(0), z_0 = z(0),\ u_0 = u(0) = \dot{z}(0) = z_1$ .The fourth-order subsystem from the Lorenz system will be called fourth-order shortened Lorenz system. For this system (6) in [23] is done dynamical analysis. When the system condition

$$m_1, m_2 = 2m_1 \in \mathbb{R} \qquad (*)$$

is true, we will speak for a special case of a fourth-order shortened Lorenz system (6).

**Remark 1.** By the notation $r^*$ in the fourth-order shortened Lorenz system (6), the parameter *r* from Modified Lorenz system (5) has been replaced. Because the notation *r* will be used with another meaning in this paper.

The explicit solutions of the Modified Lorenz system (5) from [21] can be used for solving of the fourth-order shortened Lorenz system (6). But, these solutions are complex for use. In this paper under specific conditions with proving integrability of a subclass of differential equations from the extended class linear differential equations [24] which are presented in [1], we will be offered simpler obtaining of a particular solution for a special case of the fourth-order shortened Lorenz system (6).

**Remark 2.** In the paper [25] in the same way, a particular solution of the third-order shortened Lorenz system via integrability of a class of differential equations is already offered. Therefore, this paper will follow the already published paper [25].

The integrability of this extended class of differential equations gives us explicit formulas for one particular solution. A subclass from this extended class of differential equations will be obtained, which will be used for solving the fourth-order shortened Lorenz system (6).

This paper gives only theoretical access without examples, which is small, but an essential contribution to solving differential equations.

## 2. MAIN RESULTS

In this part, the subclass from the extended class of second-order linear differential equations of B.S. Popov is obtained, which can be used for solving of the fourth-order shortened Lorenz system (6). For this goal, the following Theorem 2 will be proved.

**Theorem 2.** Let the differential equation

$$\ddot{z} + \beta\dot{z} + (A_1 e^{2t} + B_1 e^t + C_1)z = 0, \quad \beta, A_1, B_1, C_1 \in \mathbb{R} \qquad (7)$$

is given. The differential equation (7) is integrable if there exists an integer $n \in \mathbb{Z}$ that satisfies the condition

$$B_1 + (\mp\sqrt{-A_1})[2n + 1 + (\mp\sqrt{\beta^2 - 4C_1})] = 0, A_1 < 0 \quad (8)$$

In doing so, the differential equation (7) has a particular solution which is given by the formula

$$z_p(t) = e^{-\int (re^t + s)dt} y_p(e^t) \qquad (9)$$

where

$$y_p(x) = P_n(x) = x^{1-E} e^{-Dx} [x^{n-1+E} e^{Dx}]^{(n)} \qquad (10)$$

if $n \in \mathbb{N}$
or

$$y_p(x) = [x^{k+1-E} e^{-Dx}]^{(k)}, \qquad (11)$$

if $n \in \mathbb{Z}^-, k = -(n+1) \in \mathbb{N}$.
By the relations

$$\begin{aligned} &D = 2(\mp\sqrt{-A_1}), \qquad E = 1 + (\mp\sqrt{\beta^2 - 4C_1}), \qquad r = \pm\sqrt{-A_1}, \\ &s = \frac{1}{2}(\beta \pm \sqrt{\beta^2 - 4C_1}), \qquad F = B_1 + (\mp\sqrt{-A_1})[1 + (\mp\sqrt{\beta^2 - 4C_1})] \end{aligned} \qquad (12)$$

the coefficients in the formulas (9), (10) and (11) are obtained.

*Proof.* Let us consider the differential equation

$$xy'' + (Dx + E)y' + Fy = 0, \quad D, E, F \in \mathbb{R} \qquad (13)$$

where

$$y = y(x), y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}.$$

By the substitution

$$x = e^t \qquad (14)$$

the differential equation (13) can be written as

$$\ddot{y} + [De^t + E - 1]\dot{y} + Fe^t y = 0 \quad (15)$$

where

$$y = y(x), \dot{y} = \frac{dy}{dt}, \ddot{y} = \frac{d^2y}{dt^2}.$$

By the substitution

$$y(t) = e^{\int (re^t + s)dt} z(t), \quad r, s \in \mathbb{R} \qquad (16)$$

the equation (15) is transformed in the differential equation

$$\begin{aligned} &\ddot{z} + [(2r + D)e^t + 2s + E - 1]\dot{z} + [(r^2 + rD)e^{2t} + (2rs + rE + sD + F)e^t \\ &\quad + s^2 + sE - s]z = 0 \end{aligned} \qquad (17)$$

where

$$z = z(t), \dot{z} = \frac{dz}{dt}, \ddot{z} = \frac{d^2z}{dt^2}.$$

The equation (7) is equal of the equation (17) if the following relations

$$
\begin{aligned}
&2r + D = 0 \\
&2s + E - 1 = \beta \\
&r^2 + rD = A_1 \qquad\qquad (18) \\
&2rs + rE + Ds + F = B_1 \\
&s^2 + sE - s = C_1
\end{aligned}
$$

are satisfied. From (18), the relations (12) are obtained.

By using the Theorem 1, the equation (13) is integrable if there exists an integer $n \in \mathbb{Z}$ that satisfies the condition

$$nD + F = 0 \qquad (19)$$

In accordance with the relations (12), the condition (19) is equal to the condition (8). By using the formulas (3) and (4) from Theorem 1 applied to the equation (13), the formulas (10) and (11) are obtained. Finally, in accordance with the substitutions (14) and (16), the formula (9) is obtained. □

**Remark 3.** In connections (12) the sign before the roots is equal to the sign before the roots the condition (8).

By Theorem 3, the last two equations of the fourth-order shortened Lorenz system (6) for given initial values offered a particular solution.

**Theorem 3.** The last two equations of the fourth-order shortened Lorenz system (6) are transformed in a second-order linear homogeneous differential equation with the constant coefficients

$$\ddot{z} - (m_1 + m_2)\dot{z} + m_1 m_2 z = 0 \qquad (20).$$

The differential equation (20) with the initial values $z_0 = z(0),\ z_1 = \dot{z}(0)$, and the condition (*) for $m_1 = m$ has the particular solution

$$z_p(t) = We^{mt} + Le^{2mt}, W = \frac{2mz_0 - z_1}{m}, L = \frac{z_1 - mz_0}{m}. \qquad (21)$$

*Proof.* By help of the fourth equation and differentiation of the third equation of the fourth-order shortened Lorenz system (6), a second-order linear homogeneous differential equation with the constant coefficients (20) is obtained. The characteristic equation of the differential equation (20) is

$$m^2 - (m_1 + m_2)m + m_1 m_2 = 0$$

by solutions $m_1, m_2$. The general solution of the differential equation (20) is

$$z(t) = We^{m_1 t} + Le^{m_2 t}, \quad W, L = \mathrm{c}onst.$$

Nothing is lost from generality, if we assume that $m_1 = m$. For the initial values $z_0 = z(0),\ z_1 = \dot{z}(0)$ and the condition $m_1 = m,\ m_2 = 2m_1 = 2m$, the particular solution

$$z_p(t) = We^{mt} + Le^{2mt}, \quad W = \frac{2mz_0 - z_1}{m}, L = \frac{z_1 - mz_0}{m}$$

is obtained. □

By Theorem 4, the first two equations of the fourth-order shortened Lorenz system (6) in a second-order linear differential equation are transformed.

**Theorem 4.** The first two equations of the fourth-order shortened Lorenz system (6) are transformed in a second-order linear differential equation

$$\ddot{x} + (\sigma+1)\dot{x} + \sigma(1 - r^* + We^{mt} + Le^{2mt})x = 0, \quad \sigma, r^* > 0, \quad m \in \mathbb{R} \quad (22)$$

where

$$x = x(t), \dot{x} = \frac{dx}{dt}, \ddot{x} = \frac{d^2x}{dt^2}, W = \frac{2mz_0 - z_1}{m}, L = \frac{z_1 - mz_0}{m}.$$

*Proof.* By help of the second equation and differentiation of the first equation of the fourth-order shortened Lorenz system (6), a second-order linear differential equation

$$\ddot{x} + (\sigma+1)\dot{x} + \sigma(1 - r^* + z_p(t))x = 0, \quad \sigma, r^* > 0$$

is obtained, where

$$x = x(t), \dot{x} = \frac{dx}{dt}, \ddot{x} = \frac{d^2x}{dt^2}$$

By using of the particular solution (21), the second-order linear differential equation (22) is obtained. □

The condition for integrability of the differential equation (22) in Theorem 4 is given by Theorem 5. In accordance with the formulas of Theorem 2, one particular solution is obtained by Theorem 5.

**Theorem 5.** Let the differential equation (22) is given. The differential equation (22) is integrable if there exists an integer $n \in \mathbb{Z}$ that satisfies the condition

$$\sigma W + (\mp\sqrt{-\sigma L})[(2n+1)m + (\mp\sqrt{(\sigma-1)^2 + 4\sigma r^*}) = 0, L < 0, m \in \mathbb{R}, \quad (23)$$

where $W = \dfrac{2mz_0 - z_1}{m}$, $L = \dfrac{z_1 - mz_0}{m}$, $z_0, z_1 \in \mathbb{R}$.

In doing so, the differential equation (20) has a particular solution which is given by the formula

$$x_p(t^*) = e^{-\int (re^{t^*}+s)dt^*} y_p(e^{t^*}) \qquad (24)$$

where

$$y_p(x) = P_n(x) = x^{1-E} e^{-Dx} [x^{n-1+E} e^{Dx}]^{(n)}$$

if $n \in \mathbb{N}$
or

$$y_p(x) = [x^{k+1-E} e^{-Dx}]^{(k)}$$

if $n \in \mathbb{Z}^-, k = -(n+1) \in \mathbb{N}$ for $t^* = mt$.
By the relations

$$D = \frac{2}{m}\left(\mp\sqrt{-\sigma L}\right), \quad E = 1 \mp \frac{1}{m}\sqrt{(\sigma-1)^2 + 4\sigma r^*}, \quad r = \frac{1}{m}\left(\pm\sqrt{-\sigma L}\right),$$

$$s = \frac{1}{2m}\left[(\sigma+1) \pm \sqrt{(\sigma-1)^2 + 4\sigma r^*}\right],$$

$$F = \frac{1}{m}\left[\frac{1}{m}\sigma W + (\mp\sqrt{-\sigma L})(1 \mp \frac{1}{m}\sqrt{(\sigma-1)^2 + 4\sigma r^*}\right]$$

the coefficients $D, r, s, E, F$ are obtained.

*Proof.* By the substitution

$$mt = t^* \qquad (25)$$

the equation (20) are transformed in the differential equation

$$\ddot{x}(t^*) + \frac{1}{m}(\sigma+1)\dot{x}(t^*) + \frac{1}{m^2}\sigma(Le^{2t^*} + We^{t^*} + 1 - r^*)x(t^*) = 0 \qquad (26)$$

where

$$x = x(t), \dot{x} = \frac{dx}{dt}, \ddot{x} = \frac{d^2x}{dt^2}.$$

The differential equation (26) is equal by the equation (7), if the relations

$$\beta = \frac{1}{m}(\sigma+1), A_1 = \frac{1}{m^2}\sigma L, B_1 = \frac{1}{m^2}\sigma W, C_1 = \frac{1}{m^2}\sigma(1-r^*)$$

are valid.
The condition (8) of Theorem 2 applied to the equation (26) is the condition (23). By using the formula (9) of Theorem 2, formula (24) is obtained. In accordance with the substitution (25), the formula of one particular solution is obtained. □

A particular solution $(x_p(t), y_p(t), z_p(t))$ of a special case of the fourth-order shortened Lorenz system (6) is obtained by the following Theorem 6.

**Theorem 6.** A particular solution $(x_p(t), y_p(t), z_p(t))$ of a special case of the fourth-order shortened Lorenz system (6) when the condition (23) is satisfied is obtained as follows:

- for $x_p(t)$ with the formula (24);
- $y_p(t) = \frac{1}{\sigma}\dot{x}_p(t) + x_p(t)$ where $\dot{x}_p(t) = \frac{dx_p}{dt}$;
- for $z_p(t)$ with the formula (21).

*Proof.* It is clear that a particular solution $(x_p(t), y_p(t), z_p(t))$ of the fourth-order shortened Lorenz system (6) can be found in the condition (23) of Theorem 5 is satisfied. The particular solution is obtained by using the formulas for one particular solution (24) for one particular solution from Theorem 5 for $x_p(t)$, the formula (21) from Theorem 3 for $z_p(t)$ and by using the first equation of the fourth-order shortened Lorenz system (6) with $y_p(t) = \frac{1}{\sigma}\dot{x}_p(t) + x_p(t)$, where $\dot{x}_p(t) = \frac{dx_p}{dt}$. □

## 3. CONCLUSIONS

In this paper for the special case of the fourth-order shortened Lorenz system (6), a way for theoretically obtaining one particular solution was presented. We speak for a finding of a particular solution for a small class of systems of differential equations, but solving such a nonlinear system is complex even with a computer.

It would be good to be given an appropriate example with concrete initial values and its geometrical visualization. But, the choice of such an example is a difficult and complex process even with a computer.

Therefore, this paper gives only theoretical access, which is an essential contribution to solving differential equations.

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