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INVERTIBILITY OF LINEAR COMBINATIONS OF K-POTENT MATRICES

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Abstract. We study the problem of the invertibility of linear combinations of two or three k-potent matrices under various conditions. In these cases, we give explicit formulae of their inverses.

1. INTRODUCTION

Let $\mathbb{C}^{m\times n}$ denote the set of all $m\times n$ complex matrices. Specially, let $\mathbb{C}^{n\times n}$ denote the set of all $n\times n$ square complex matrices. The symbols $R(A)$ and $N(A)$ denote the range (column space) and the null space of a matrix A, respectively, while $r(A)$ is rank of A. $\mathbb{C}_r^{n\times n}$ is symbol of the set of all $n\times n$ matrices with rank r. Also, I_n denotes the identity matrix of order n. We say that integers k and l are congruent modulo the positive integer m, and we use notation $k\equiv l\ (mod\ m)$, if m divide $k-l$.
In this paper, we deal with *k*-potent matrices, where *k* is a positive integer greater than one. This type of matrices is defined as follows.

Definition 1. ([4]) *A matrix* $A\epsilon\mathbb{C}^{n\times n}$ *is k-potent if* $A^k=A$, *where* k *is a positive integer greater than one.*

Any *k*-potent matrix is group invertible. For $A\epsilon\mathbb{C}^{n\times n}$, the group inverse of A is the unique, if it exists (see [2]), matrix $A^{\#}\epsilon\mathbb{C}^{n\times n}$ such that:

$$A=AA^{\#}A, A^{\#}=A^{\#}AA^{\#}, AA^{\#}=A^{\#}A.$$

Thus, if $A\epsilon\mathbb{C}^{n\times n}$ is a *k*-potent matrix, then $A^{\#}=A^{k-2}$.

The research dealing with *k*-potent matrices is quite extensive (see [1], [4]-[6], [8], [9]) because they have a wide application, for example in statistics (see [7], [10]). A particularly interesting research topic related to *k*-potent matrices is the invertibility of a linear combination of *k*-potent matrices. In this paper, we study the invertibility of linear combinations c_1A+c_2B and $c_1A+c_2B+c_3C$, where A,B,C are k-potent matrices and c_1,c_2,c_3 are nonzero complex numbers. Also, we give some formulae for $(c_1A+c_2B)^{-1}$ and $(c_1A+c_2B+c_3C)^{-1}$ under various conditions.

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2. INVERTIBILITY OF A LINEAR COMBINATION OF TWO K-POTENT MATRICES

Recently, there has been interest in investigating the invertibility of a linear combination of two *k*-potent matrices. In [3], J. Benitez, X. Liu and T. Zhu proved the following results.

Theorem 1. [3] *Let* $A, B \in \mathbb{C}^{n\times n}$ *be two k-potent matrices such that* $A^{k-1}B = B^{k-1}A$ *or* $BA^{k-1} = AB^{k-1}$. *If a linear combination* $d_1A + d_2B$ *is nonsingular for some* $d_1, d_2 \in \mathbb{C} \setminus \{0\}$ *satisfying* $d_1 + d_2 \neq 0$, *then* $c_1A + c_2B$ *is nonsingular for all* $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ *satisfying* $c_1 + c_2 \neq 0$.

Theorem 2. **[3]** *Let* $A, B \in \mathbb{C}^{n\times n}$ *be two k-potent matrices such that* $I_n - A^{k-1}B^{k-2}$ *is nonsingular. If there exist* $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ *such* $c_1A + c_2B$ *is nonsingular, then* $A - B$ *is also nonsingular.*

Theorem 3. [3] *Let* $A, B \in \mathbb{C}^{n\times n}$ *be two commuting k-potent matrices. If there exists* $a \in \mathbb{C} \setminus \{0\}$ *such that* $A + aB$ *is nonsingular, then* $c_1A + c_2B$ *and* $c_1I_n + c_2AB$ *are nonsingular for all* $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ *with* $\frac{-c_1}{c_2} \notin \sqrt[k-1]{1}$.

Theorem 4. [3] *Let* $A, B \in \mathbb{C}^{n\times n}$ *be two k-potent matrices, and let* $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are equivalent:*

(i) $c_1AB^{k-1} + c_2BA^{k-1}$ *is nonsingular.*
(ii) $c_1B^{k-1}A + c_2A^{k-1}B$ *is nonsingular.*
(iii) $c_1A + c_2B$ *and* $I_n - A^{k-1} - B^{k-1}$ *are nonsingular.*

Theorem 5. [3] *Let* $A, B \in \mathbb{C}^{n\times n}$ *be two k-potent matrices such that* $A^{k-1}B = B^{k-1}A$, *and let* $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. *If* A *or* B *are nonsingular, then* $c_1A + c_2B$ *are nonsingular if and only if* $c_1 + c_2 \neq 0$. *In this case,*

(i) *If A is nonsingular, then*

$$(c_1 + c_2)(c_1A + c_2B)^{-1} = A^{-1} + c_2c_1^{-1}A^{-1}(I_n - B^{k-1}).$$

(ii) *If B is nonsingular, then*

$$(c_1 + c_2)(c_1A + c_2B)^{-1} = B^{-1} + c_1c_2^{-1}B^{-1}(I_n - A^{k-1}).$$

Theorem 6. [3] *Let* $A, B \in \mathbb{C}^{n\times n}$ *be two k-potent matrices such that* $AB = 0$, *and let* $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. *Then* $N(c_1A + c_2B) = N(A + B)$ *and* $R(c_1A + c_2B) =$

$R(A+B)$. *In particular,* c_1A+c_2B *is nonsingular if and only if* $A+B$ *is nonsingular, and in this case, we have*

$$(c_1A+c_2B)^{-1}=c_1^{-1}(A+B)^{-1}+(c_2^{-1}-c_1^{-1})B^{k-2}\left(I_n-A^{k-1}\right).$$

However, the invertibility of a linear combination can be studied under other conditions.
Let $A,B\epsilon\mathbb{C}^{n\times n}$ be two k-potent matrices for some natural $k>1$. Since $A\epsilon\mathbb{C}^{n\times n}, r(A)=r$ is k-potent, this matrix can be written as:

$$A=U\begin{bmatrix}K & 0\\ 0 & 0\end{bmatrix}U^{-1}, \quad (1)$$

where $U\epsilon\mathbb{C}^{n\times n}$ is nonsingular, $K=diag(\lambda_1,\dots,\lambda_r)$, $\lambda_i^{K-1}=1$ for $i=1,\dots,r$. Obviously, $K\epsilon\mathbb{C}^{r\times r}$ is nonsingular and $K^{k-1}=I_r$. Furthermore, we can write $B\epsilon\mathbb{C}^{n\times n}$ as follows:

$$B=U\begin{bmatrix}B_1 & B_2\\ B_3 & B_4\end{bmatrix}U^{-1}, \quad (2)$$

where $B_1=\mathbb{C}^{r\times r}$ and $B_4\epsilon\mathbb{C}^{(n-r)\times(n-r)}$.

Theorem 7. *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices such that* $AB=0=BA$, *and let* $c_1,c_2\epsilon\mathbb{C}\setminus\{0\}$. *Then* c_1A+c_2B *is nonsingular if and only if* $A^{k-1}+\left(I_n-A^{k-1}\right)B$ *is nonsingular. Furthermore,*

$$(c_1A+c_2B)^{-1}=c_1^{-1}A^{k-2}+c_2^{-1}\left(I_n-A^{k-1}\right)B^{k-2}.$$

Proof. Let $A\epsilon\mathbb{C}_r^{n\times n}$ be of the form (1). Since $AB=0=BA$, then B has the form:

$$B=U\begin{bmatrix}0 & 0\\ 0 & B_4\end{bmatrix}U^{-1}, \quad (3)$$

where $B_4\epsilon\mathbb{C}^{(n-r)\times(n-r)}$. From $B^k=B$, it follows that $B_4^k=B_4$, i.e. B_4 is k-potent. In addition,

$$c_1A+c_2B=U\begin{bmatrix}c_1K & 0\\ 0 & c_2B_4\end{bmatrix}U^{-1}.$$

Based on the invertibility of K, we conclude that c_1A+c_2B is nonsingular if and only if B_4 is nonsingular. Since

$$A^{k-1}+\left(I_n-A^{k-1}\right)B=U\begin{bmatrix}I_r & 0\\ 0 & B_4\end{bmatrix}U^{-1},$$

we have that c_1A+c_2B is nonsingular if and only if $A^{k-1}+\left(I_n-A^{k-1}\right)B$ is nonsingular. Furthermore,

$$(c_1A+c_2B)^{-1}=U\begin{bmatrix}c_1^{-1}K^{-1} & 0\\ 0 & c_2^{-1}B_4^{-1}\end{bmatrix}U^{-1}=U\begin{bmatrix}c_1^{-1}K^{k-2} & 0\\ 0 & c_2^{-1}B_4^{k-2}\end{bmatrix}U^{-1}$$
$$=c_1^{-1}A^{k-2}+c_2^{-1}(I_n-A^{k-1})B^{k-2}. \quad \square$$

Corollary 1. *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices such that* $AB=0=BA$*, and let* $c_1,c_2\epsilon\mathbb{C}\setminus\{0\}$*. Then,* c_1A+c_2B *is nonsingular if and only if* $A+B$ *is nonsingular.*

Beside forms (1) and (2), the following fact: *If* $E,F\epsilon\mathbb{C}^{n\times n}$ *and* $EF=FE$*, then*

$$E^k+(-1)^{k+1}F^k=(E+F)\sum_{i=0}^{k-1}(-1)^i E^{k-1-i}F^i,\ k\epsilon N,\ k>1,$$

is very useful for next results. First, note that $A^l=A^s$, where A is a *k*-potent matrix and $l\equiv s(mod(k-1))$.

Theorem 8. [11] *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices, and let* $c_1,c_2\epsilon\mathbb{C}\setminus\{0\}$.

(i) *If* $AB=A^s$, $s\epsilon\{0,1,3,\dots,k-2\}$, *and* $c_1^{k-1}+(-1)^kc_2^{k-1}\neq 0$, *then* c_1A+c_2B *is nonsingular if and only if* $A^{k-1}+B(I_n-A^{k-1})$ *is nonsingular. Furthermore,*

$$(c_1A+c_2B)^{-1}=A_1-B^{k-2}(I_n-A^{k-1})^2BA^{k-1}A_1+\frac{1}{c_2}B^{k-2}(I_n-A^{k-1}),$$

where: $A_1=\frac{1}{c_1^{k-1}+(-1)c_2^{k-1}}\sum_{i=0}^{k-2}(-1)^ic_1^{k-2-i}c_2^iA^{k-2-i+(s-1)i}$.

(ii) *If* $AB=A^2$ *and* $c_1+c_2\neq 0$, *then* c_1A+c_2B *is nonsingular if and only if* $A^{k-1}+B(I_n-A^{k-1})$ *is nonsingular. Furthermore,*

$$(c_1A+c_2B)^{-1}=\frac{1}{c_1+c_2}A^{k-2}-\frac{1}{c_1+c_2}B^{k-2}(I_n-A^{k-1})^2BA^{k-1}A^{k-2}+\frac{1}{c_2}B^{k-2}(I_n-A^{k-1}).$$

Corollary 2. *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices, and let* $c_1,c_2\epsilon\mathbb{C}\setminus\{0\}$.

(i) *If* $AB=A^s$, $s\epsilon\{0,1,3,\dots,k-2\}$, *and* $c_1^{k-1}+(-1)^kc_2^{k-1}\neq 0$, *then* c_1A+c_2B *is nonsingular if and only if* $A+B$ *is nonsingular.*

(ii) *If* $AB=A^2$ *and* $c_1+c_2\neq 0$, *then* c_1A+c_2B *is nonsingular if and only if* $A+B$ *is nonsingular.*

Theorem 9. [11] *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices, and let* $c_1, c_2\epsilon\mathbb{C}\setminus\{0\}$.

(*i*) *If* $BA = A^s$, $s\epsilon\{0,1,3,\dots,k-2\}$, *and* $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, *then* $c_1A + c_2B$ *is nonsingular if and only if* $A^{k-1} + (I_n - A^{k-1})B$ *is nonsingular. Furthermore,*

$$(c_1A + c_2B)^{-1} = A_1 - A_1A^{k-1}B\left(I_n - A^{k-1}\right)^2B^{k-2} + \frac{1}{c_2}\left(I_n - A^{k-1}\right)B^{k-2}, \tag{6}$$

where: $A_1 = \frac{1}{c_1^{k-1}+(-1)^k c_2^{k-1}}\sum_{i=0}^{k-2}(-1)^i c_1^{k-2-i}c_2^i A^{k-2-i+(s-1)i}$.

(*ii*) *If* $BA = A^2$ *and* $c_1 + c_2 \neq 0$, *then* $c_1A + c_2B$ *is nonsingular if and only if* $A^{k-1} + (I_n - A^{k-1})B$ *is nonsingular. Furthermore,*

$$(c_1A + c_2B)^{-1} = \frac{1}{c_1+c_2}A^{k-2} - \frac{1}{c_1+c_2}A^{k-2}A^{k-1}B\left(I_n - A^{k-1}\right)^2B^{k-2} + \frac{1}{c_2}\left(I_n - A^{k-1}\right)B^{k-2}.$$

Corollary 3. *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices, and let* $c_1, c_2\epsilon\mathbb{C}\setminus\{0\}$.

(*i*) *If* $BA = A^s$, $s\epsilon\{0,1,3,\dots,k-2\}$, *and* $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, *then* $c_1A + c_2B$ *is nonsingular if and only if* $A + B$ *is nonsingular.*

(*ii*) *If* $BA = A^2$ *and* $c_1 + c_2 \neq 0$, *then* $c_1A + c_2B$ *is nonsingular if and only if* $A + B$ *is nonsingular.*

Theorem 10. [11] *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be commuting k-potent matrices, and let* $c_1, c_2\epsilon\mathbb{C}\setminus\{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. *Then* $c_1A + c_2B$ *is nonsingular if and only if* $A^{k-1} + (I_n - A^{k-1})B$ *is nonsingular. Furthermore,*

$$(c_1A + c_2B)^{-1} = \frac{1}{c_1^{k-1}+(-1)^k c_2^{k-1}}\sum_{i=0}^{k-2}(-1)^i c_1^{k-2-i}c_2^i A^{k-2-i}\left(A^{k-1}B\right)^i + \frac{1}{c_2}\left(I_n - A^{k-1}\right)B^{k-2}.$$

Corollary 4. [11] *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be commuting k-potent matrices, and let* $c_1, c_2\epsilon\mathbb{C}\setminus\{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. *Then,* $c_1A + c_2B$ *is nonsingular if and only if* $A + B$ *is nonsingular.*

Lemma 1. [11] *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *be a k-potent matrix, and let* $c_1, c_2\epsilon\mathbb{C}, c_1 \neq 0, c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. *Then* $c_1 I_n + c_2 A$ *is nonsingular and:*

$$(c_1 I_n + c_2 A\,)^{-1} = \frac{1}{c_1^{k-1}+(-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i A^i + \frac{1}{c_1}\left(I_n - A^{k-1}\right).$$

Theorem 11. [11] *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices, and let* $c_1, c_2\epsilon\mathbb{C}\setminus\{0\}, c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. *If* $AB = B$ *or* $BA = B$, *then* $c_1 A + c_2 B$ *is nonsingular if and only if* A *is nonsingular. Furthermore,*

$$(c_1 A + c_2 B)^{-1} = A^{-1}\left(\frac{1}{c_1^{k-1}+(-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i B^i + \frac{1}{c_1}\left(I_n - B^{k-1}\right)\right).$$

Corollary 5. *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices, and let* $c_1, c_2\epsilon\mathbb{C}\setminus\{0\}, c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. *If* $AB = B$ *or* $BA = B$, *then* $c_1 A + c_2 B$ *is nonsingular if and only if* $A + B$ *is nonsingular.*

3. INVERTIBILITY OF A LINEAR COMBINATION OF THREE K-POTENT MATRICES

Now, we study the invertibility of a linear combination of three k-potent matrices.

Theorem 12. *Let* $A\epsilon\mathbb{C}^{n\times n}$ *and* $B, C\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices such that* $AB = 0 = BA, AC = 0 = CA$ *and* $BC = CB$, *and let* $c_1, c_2, c_3\epsilon\mathbb{C}\setminus\{0\}$ *such that* $c_2^{k-1} + (-1)^k c_3^{k-1} \neq 0$. *Then* $c_1 A + c_2 B + c_3 C$ *is nonsingular if and only if* $A^{k-1} + (B + C)\left(I_n - A^{k-1}\right)$ *is nonsingular. Furthermore,*

$$(c_1 A + c_2 B + c_3 C)^{-1} = c_1^{-1} A^{k-2} + \left[(c_2 B + c_3 C)\left(I_n - A^{k-1}\right)\right]^{\#}. \tag{4}$$

Proof. Let $A\epsilon\mathbb{C}_r^{n\times n}$ be of the form (1). Since $AB = 0 = BA$, then B has the form (3). Suppose that $C\epsilon\mathbb{C}^{n\times n}$ has the next representation:

$$C = U\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} U^{-1},$$

where $C_1\epsilon\mathbb{C}^{r\times r}$ and $C_4\epsilon\mathbb{C}^{(n-r)\times(n-r)}$. From $AC = 0 = CA$, it follows that $C_1 = C_2 = C_3 = 0$, i.e.

$$C = U\begin{bmatrix} 0 & 0 \\ 0 & C_4 \end{bmatrix} U^{-1},$$

where $C_4\epsilon\mathbb{C}^{(n-r)\times(n-r)}$ is the *k*-potent matrix because C is the *k*-potent matrix. Now, $c_1 A + c_2 B + c_3 C$ can be represented as:

$$c_1A + c_2B + c_3C = U \begin{bmatrix} c_1K & 0 \\ 0 & c_2B_4 + c_3C_4 \end{bmatrix} U^{-1},$$

where $B_4, C_4 \epsilon \mathbb{C}^{(n-r)\times(n-r)}$ are k-potent matrices. Since $BC = CB$, then $B_4C_4 = C_4B_4$. Based on the invertibility of K, we conclude that $c_1A + c_2B + c_3C$ is nonsingular if and only if $c_2B_4 + c_3C_4$ is nonsingular for all constants $c_1, c_2, c_3 \epsilon \mathbb{C} \setminus \{0\}$. By Corollary 4, we deduce that $c_2B_4 + c_3C_4$ is nonsingular if and only if $B_4 + C_4$ is nonsingular for all constants $c_2, c_3 \epsilon \mathbb{C} \setminus \{0\}$ such that $c_2^{k-1} + (-1)^k c_3^{k-1} \neq 0$. Furthermore,

$$A^{k-1} + (B + C)\left(I_n - A^{k-1}\right) = U \begin{bmatrix} I_r & 0 \\ 0 & B_4 + C_4 \end{bmatrix}.$$

Thus, the necessary and sufficient condition of the invertibility of $c_1A + c_2B + c_3C$ is invertibility of $A^{k-1} + (B + C)\left(I_n - A^{k-1}\right)$.
In addition, by direct computation, we get

$$\begin{aligned}(c_1A + c_2B + c_3C)^{-1} &= U \begin{bmatrix} c_1^{-1}K^{-1} & 0 \\ 0 & (c_2B_4 + c_3C_4)^{-1} \end{bmatrix} U^{-1} \\ &= U \begin{bmatrix} c_1^{-1}K^{k-2} & 0 \\ 0 & (c_2B_4 + c_3C_4)^{-1} \end{bmatrix} U^{-1}.\end{aligned}$$

Note that $(c_2B + c_3C)\left(I_n - A^{k-1}\right) = U \begin{bmatrix} 0 & 0 \\ 0 & (c_2B_4 + c_3C_4)^{-1} \end{bmatrix} U^{-1}.$

Hence, the formula (4) holds. □

Corollary 6. *Let* $A \epsilon \mathbb{C}^{n\times n}$, *and* $B, C \epsilon \mathbb{C}^{n\times n}$ *be k-potent matrices such that* $AB = 0 = BA$, $AC = 0 = CA$ *and* $BC = CB$, *and let* $c_1, c_2, c_3 \epsilon \mathbb{C} \setminus \{0\}$ *such that* $c_2^{k-1} + (-1)^k c_3^{k-1} \neq 0$. *Then,* $c_1A + c_2B + c_3C$ *is nonsingular if and only if* $A + B + C$ *is nonsingular.*

The next results are proved in [11].

Theorem 13. [11] *Let* $A \epsilon \mathbb{C}_r^{n\times n}$ *and* $B, C \epsilon \mathbb{C}^{n\times n}$ *be k-potent matrices such that* $AC = 0 = CA$ *and* $BC = CB$, *and let* $c_1, c_2, c_3 \epsilon \mathbb{C} \setminus \{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$.

(*i*) *If* $AB = A^s$, $s \epsilon \{0, 1, 3, \dots, k-2\}$, *and* $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, *then* $c_1A + c_2B + c_3C$ *is nonsingular if and only if* $A^{k-1} + (B + C)\left(I_n - A^{k-1}\right)$ *is nonsingular. Furthermore,*

$$(c_1A + c_2B + c_3C\,)^{-1} = A_1 - \tfrac{1}{c_2}\left[(c_2B + c_3C)\left(I_n - A^{k-1}\right)\right]^{\#},$$

where:

$$A_1 = \frac{1}{c_1^{k-1} + (-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i A^{k-2-i+(s-1)i}.$$

(ii) *If* $AB = A^2$, *then* $c_1A + c_2B + c_3C$ *is nonsingular if and only if* $A^{k-1} + (B + C)\left(I_n - A^{k-1}\right)$ *is nonsingular. Furthermore,*

$$(c_1A + c_2B + C_3C)^{-1} = \tfrac{1}{c_1+c_2}A^{k-2} - \tfrac{1}{c_2(c_1+c_2)}\left[(c_2B + c_3C)\left(I_n - A^{k-1}\right)\right]^{\#}\left(I_n - A^{k-1}\right)BA^{k-1}A^{k-2} + \left[(C_2B + C_3C)\left(I_n - A^{k-1}\right)\right]^{\#}.$$

Corollary 7. *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B, C\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices such that* $AC = 0 = CA$ *and* $BC = CB$, *and let* $c_1, c_2, c_3\epsilon\mathbb{C}\setminus\{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$.

(i) *If* $AB = A^s$, $s\epsilon\{0, 1, 3, \dots, k-2\}$, *and* $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, *then* $c_1A + c_2B + c_3C$ *is nonsingular if and only if* $A + B + C$ *is nonsingular.*

(ii) *If* $AB = A^2$, *then* $c_1A + c_2B + c_3C$ *is nonsingular if and only if* $A + B + C$ *is nonsingular.*

Theorem 14. [11] *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B, C\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices such that* $AC = 0 = CA$, *and let* $c_1, c_2, c_3\epsilon\mathbb{C}\setminus\{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. *If* $AB = B$ *or* $BA = B$, *then* $c_1A + c_2B + c_3C$ *is nonsingular if and only if* $A^{k-1} + C\left(I_n - A^{k-1}\right)$ *is nonsingular. Furthermore,*

$$(c_1A + c_2B + c_3C)^{-1} = A^{k-2}\left(\tfrac{1}{c_1^{k-1}+(-1)^k c_2^{k-1}}\textstyle\sum_{i=0}^{k-2}(-1)^i c_1^{k-2-i} c_2^i B^i + \tfrac{1}{c_1}\left(I_n - B^{k-1}\right)\right) + \tfrac{1}{c_3}C^{k-2}.$$

Corollary 8. *Let Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B, C\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices such that* $AC = 0 = CA$, *and let* $c_1, c_2, c_3\epsilon\mathbb{C}\setminus\{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. *If* $AB = B$ *or* $BA = B$, *then* $c_1A + c_2B + c_3C$ *is nonsingular if and only if* $A + B + C$ *is nonsingular.*

Theorem 15. [11] *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B, C\epsilon\mathbb{C}^{n\times n}$ *be commuting k-potent matrices such that* $AC = 0 = CA$, *and let* $c_1, c_2, c_3\epsilon\mathbb{C}\setminus\{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, *and* $c_2^{k-1} + (-1)^k c_3^{k-1} \neq 0$. *Then,* $c_1A + c_2B + c_3C$ *is nonsingular if and only if* $A^{k-1} + \left(I_n - A^{k-1}\right)(B + C)$ *is nonsingular. Furthermore,*

$$(c_1A + c_2B + c_3C)^{-1} = \frac{1}{c_1^{k-1}+(-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i A^{k-2-i} \left(A^{k-1}B\right)^i + \frac{1}{c_2}\left(I_n - A^{k-1}\right) B^{k-2} + \left[(c_2B + c_3C)\left(I_n - A^{k-1}\right)\right]^{\#}.$$

Corollary 9. *Let $A \in \mathbb{C}_r^{n\times n}$ and $B, C \in \mathbb{C}^{n\times n}$ be commuting k-potent matrices such that $AC = 0 = CA$, and let $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, and $c_2^{k-1} + (-1)^k c_3^{k-1} \neq 0$. Then, $c_1A + c_2B + c_3C$ is nonsingular if and only if $A + B + C$ is nonsingular.*

4. CONCLUSIONS

The paper provided investigations of the invertibility of linear combinations of k-potent complex matrices. Several new properties of the invertibility of k-potent matrices are identtified. Furthermore, some results in the literature are restablished. The most important conclusion is that the invertibility of the linear combination of k-potent matrices is equivalent to the invertibility of the sum of given matrices. Thus, the invertibility of the linear combination of two or three k-potent matrices is independent of the choice of the nonzero complex constants.

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