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DEFORMED SPHERICAL CURVES

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Abstract. This paper is devoted to the study of spherical curves in Euclidean three-dimensional space using the theory of infinitesimal bending. Some interesting infinitesimal bending fields are obtained and discussed. In particular, the infinitesimal bending of a spherical curve such that all bent curves are approximately on the initial sphere (with a given precision) is studied. Some examples are analyzed and graphically presented.

1. Introduction

A spherical curve is a curve traced on a sphere. The curvature-to-torsion ratio completely describes a spherical curve in the sense that certain relations between curvature and torsion are necessary and sufficient for the space curve to lie on the sphere.

Deformations of spherical curves represent an interesting field of research. Some of the recent results in this regard are in the papers [3], [4], [6], [14]. A special type of small deformation is the so-called infinitesimal bending. A concept of infinitesimal bending first appeared in the description of the deformation of surfaces in three-dimensional Euclidean space, and then further extended to the curves and the manifolds. The theory of infinitesimal bending deals with vector fields and quantities associated with them, defined at the points of observed geometric objects and satisfying deformation equations. Under infinitesimal bending, the length of the arc is invariant with appropriate precision. In other words, in the initial moment of a deformation, the arc length is stationary, i.e. the initial velocity of its change is zero. Some papers related to infinitesimal bending of curves, knots and surfaces are [1], [2], [5], [7-14]. In this paper, we pay attention to spherical curves and their behavior during this type of deformation.

The paper is organized as follows: In Sec. 2 preliminary results and notation regarding infinitesimal bending of curves are presented. In Sec. 3 spherical curves and corresponding infinitesimal bending fields are studied. We suppose that the curves suffer a small deformation such that they remain on the same sphere with a given precision. In Sec. 4 and Sec. 5, respectively, Viviani's curve

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and spherical curve with constant slope (spherical helix) are discussed both analytically and graphically.

2. INFINITESIMAL BENDING

Let us consider a regular curve

$$C: \mathbf{r} = \mathbf{r}(t), \quad t \in J \subseteq \mathbb{R}^3 \tag{1}$$

of a class C^k, $k \geq 3$, included in a family of the curves

$$C_\epsilon: \mathbf{r}_\epsilon(t) = \mathbf{r}(t) + \epsilon \mathbf{z}(t) \tag{2}$$

where $\epsilon \geq 0, \epsilon \to 0$, and we get C for $\epsilon = 0$ ($C = C_0$).

Definition 1. *A continuous one-parameter family of curves C_ϵ, given by Eq.* (2), *is called an* **infinitesimal deformation** *of the curve C, given by Eq.* (1). *A field $\boldsymbol{z}(t) \in C^k, k \geq 3,$ is a vector function defined in the points of C called a* **deformation field**.

Definition 2. [2] *An infinitesimal deformation C_ϵ is an* **infinitesimal bending** of the curve C *if*

$$ds_\epsilon^2 - ds^2 = o(\epsilon). \tag{3}$$

The field $\mathbf{z}(t)$ *is the* **infinitesimal bending field** *of the curve C.*

According to Def. 2, the next theorem states.

Theorem 1. [2] *Necessary and sufficient condition for the curves C_ϵ to be infinitesimal bending of the curve C is to be valid*

$$d\mathbf{r} \cdot d\mathbf{z} = 0. \tag{4}$$

If infinitesimal bending is reduced to rigid motion of the curve, without internal deformations, we say it is **trivial** infinitesimal bending. The corresponding bending field is also called trivial.

Based on [11] we have the following theorem.

Theorem 2. *Under infinitesimal bending of curves each line element gets non-negative addition, which is the infinitesimal value of the order higher than the first with respect to ϵ, i. e.*

$$ds_\epsilon - ds \geq o(\epsilon).$$

The following theorem is related to determination of the infinitesimal bending field of a curve C.

Theorem 3. [12] *The infinitesimal bending field for the curve C is*

$$\mathbf{z}(t) = \int [p(t)\mathbf{n}_1(t) + q(t)\boldsymbol{n}_2(t)]\, dt, \tag{5}$$

where $p(t)$ and $q(t)$ are arbitrary integrable functions, and vectors $\mathbf{n}_1(t)$ and $\mathbf{n}_2(t)$ are respectively unit principal normal and binormal vector fields of the curve C.

3. SPHERICAL CURVES UNDER INFINITESIMAL BENDING

Let

$$C: \mathbf{r}(t) = (a\cos u(t)\cos v(t)\,, a\sin u(t)\cos v(t)\,, a\sin v(t)) \tag{6}$$

be a spherical curve on the sphere

$$S: \mathbf{r}(u,v) = (a\cos u\cos v\,, a\sin u\cos v\,, a\sin v),$$

with radius a and

$$C_\epsilon:\ \mathbf{r}_\epsilon(t) = \mathbf{r}(t) + \epsilon\mathbf{z}(t)$$

be an infinitesimal deformation of C, where $\mathbf{z}(t)$ is a deformation field. The following theorem gives the explicit expression for the field $\mathbf{z}(t)$ to be infinitesimal bending field for the spherical curve C.

Theorem 4. Let $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$ be the vector field such that

$$\begin{aligned} z_1(t) &= \int \cos u\,(t)\cos v(t)\,dt\ + c_1, \\ z_2(t) &= \int \sin u(t)\cos v(t)dt + c_2, \\ z_3(t) &= \int \sin v(t)dt + c_3, \end{aligned} \tag{7}$$

c_1, c_2, c_3 are constants. Then $\mathbf{z}(t)$ is the infinitesimal bending field for the spherical curve C given by Eq. (6).

Proof. We have

$$\dot{\mathbf{r}}(t) = \begin{cases} -a\dot{u}(t)\sin u(t)\cos v(t) - a\dot{v}(t)\cos u(t)\sin v(t) \\ a\dot{u}(t)\cos u(t)\cos v(t) - a\dot{v}(t)\sin u(t)\sin v(t) \\ a\dot{v}(t)\cos v(t) \end{cases}$$

and

$$\dot{\mathbf{z}}(t) = (\cos u\,(t)\cos v(t)\,, \sin u(t)\cos v(t), \sin v(t)),$$

where 'dot' denotes derivative with respect to t. From the previous two equations we obtain $\dot{\mathbf{r}}\cdot\dot{\mathbf{z}} = 0$, which means that $\mathbf{z}$ is infinitesimal bending field.

■

Note that the previous equations do not determine all infinitesimal bending fields of the spherical curve C. If $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$, where $z_1(t), z_2(t), z_3(t)$ are arbitrary real continuous differentiable functions, then $\mathbf{z}$ can be determined from the following equation

$$\begin{aligned} &(-a\dot{u}(t)\sin u(t)\cos v(t) \\ &- a\dot{v}(t)\cos u(t)\sin v(t))\dot{z_1} + (a\dot{u}(t)\cos u(t)\cos v(t) \\ &- a\dot{v}(t)\sin u(t)\sin v(t))\dot{z_2} + a\dot{v}(t)\cos v(t)\,\dot{z_3} = 0, \end{aligned}$$

which has many solutions. Our field is obtained having in mind that $||\mathbf{r}||^2 = a^2 \to 2\mathbf{r}\cdot\dot{\mathbf{r}} = 0$ and we put $\dot{\mathbf{z}} = \frac{1}{a}\mathbf{r}$.

We posed the question whether it is possible to infinitesimally bend the spherical curve so that all bent curves are on the initial sphere. Regarding that we have the following theorem.

Theorem 5. [14] Let $C\colon \mathbf{r}\colon (t_1, t_2) \to \mathbb{R}^3$ be a curve on the sphere S. It does not exist nontrivial vector field $\mathbf{z}(t)$ so that the family of curves

$$C_\epsilon\colon\ \mathbf{r}_\epsilon(t) = \mathbf{r}(t) + \epsilon\mathbf{z}(t)$$

belongs to the sphere S.

Further, we are weakening the previous condition by requiring that the bent curves lie on a given sphere with a predetermined precision. Precisely, let us determine an infinitesimal bending field so that all bent spherical curves are on the initial sphere with a given precision, i.e. let be valid

$$F\big(x(t), y(t), z(t)\big) = 0,$$
$$F(x_\epsilon(t), y_\epsilon(t), z_\epsilon(t)) = o(\epsilon),$$

where $F(x, y, z) = 0$ is the implicit sphere equation in Cartesian coordinates x, y, z and $o(\epsilon)$ is an infinitesimal of at least second order with respect to ϵ. In connection with that we have the following theorem.

Theorem 6. Necessary and sufficient condition for the infinitesimal deformation of the spherical curve C to be on the sphere S with a given precision is that the field **z** satisfies the condition

$$\mathbf{r}\cdot\mathbf{z} = 0. \qquad (8)$$

Proof. A vector equation of a sphere S of radius a is

$$||\mathbf{r}||^2 = a^2,$$

r is the position vector of an arbitrary point on S. Let **z** be deformation field which given spherical curve leaves on the initial sphere with a given precision, i.e. let the following condition be valid

$$||\mathbf{r}_\epsilon||^2 = a^2 + o(\epsilon),$$

where $o(\epsilon)$ is an infinitesimal of at least second order with respect to ϵ. Since $\mathbf{r}_\epsilon = \mathbf{r} + \epsilon\mathbf{z}$, the previous equation reduces to

$$||\mathbf{r} + \epsilon\mathbf{z}||^2 = a^2 + o(\epsilon),$$

wherefrom we obtain

$$||\mathbf{r}||^2 + 2\epsilon\mathbf{r}\cdot\mathbf{z} + \epsilon^2||\mathbf{z}||^2 = a^2 + o(\epsilon)$$

which leads to Eq. (8). ■

In the case of the sphere it is easy to see that the unit normal $\boldsymbol{\nu}(u, v)$ satisfies $\boldsymbol{\nu}(u, v) = \frac{1}{a}\mathbf{r}(u, v)$. So, based on the condition (8), we conclude that $\mathbf{z} \perp \boldsymbol{\nu}$, i.e. **z** lies in the tangent plane of the sphere along the curve C. It means that we can present the vector $\mathbf{z}$ as the linear combination of the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ that determine tangent plane along C, i.e.

$$\mathbf{z}(t) = f(t)\mathbf{r}_u + g(t)\mathbf{r}_v,$$

where $f(t)$ and $g(t)$ are arbitrary real continuous differentiable functions. Since

$$\dot{\mathbf{z}}(t) = \dot{f}(t)\mathbf{r}_u + f(t)(\mathbf{r}_{uu}\dot{u} + \mathbf{r}_{uv}\dot{v}) + \dot{g}(t)\mathbf{r}_v + g(t)(\mathbf{r}_{vu}\dot{u} + \mathbf{r}_{vv}\dot{v})$$

and

$$\dot{\mathbf{r}}(t) = \mathbf{r}_u\dot{u} + \mathbf{r}_v\dot{v}$$

using the condition for infinitesimal bending $\dot{\mathbf{r}} \cdot \dot{\mathbf{z}} = 0$, we obtain the following equation

$$\dot{f}(t)\mathbf{r}_u \cdot \mathbf{r}_u\dot{u} + \mathbf{r}_v\dot{v} + f(t)(\mathbf{r}_{uu}\dot{u} + \mathbf{r}_{uv}\dot{v}) \cdot \mathbf{r}_u\dot{u} + \mathbf{r}_v\dot{v} + \dot{g}(t)\mathbf{r}_v \cdot \mathbf{r}_u\dot{u} + \mathbf{r}_v\dot{v}$$
$$+ g(t)(\mathbf{r}_{vu}\dot{u} + \mathbf{r}_{vv}\dot{v}) \cdot \mathbf{r}_u\dot{u} + \mathbf{r}_v\dot{v} = 0.$$

Using one of the functions $f(t)$ and $g(t)$ arbitrarily, we obtain the other from the previous linear differential equation. In this way we find infinitesimal bending of a spherical curve such that all bent curves are on the initial sphere with a given precision.

Example 1. A circle $\mathbf{r}(t) = (a\cos t, a\sin t, 0)$ has an infinitesimal bending field $\mathbf{z}(t) = (-\sin t, \cos t, f(t))$, where $f(t)$ is an arbitrary real continuous differential function. Corresponding infinitesimal bending is on the sphere with a given precision, i.e. it is valid $||\mathbf{r}_\epsilon||^2 = a^2 + \epsilon^2\left(1 + f^2(t)\right) = a^2 + o(\epsilon)$. In Fig.1 we can see deformed circle for $f(t) = \sin t \cos t$. The red color denotes original curve, and blue deformed ones for ϵ= 0.5 and ϵ= 1.

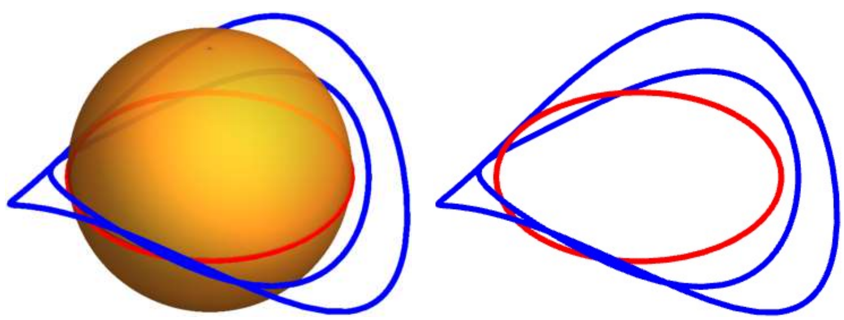

Figure 1: Circle and its infinitesimal bending for $\epsilon = 0.5$ and $\epsilon = 1$.

Based on Theorems 4 and 6 we obtain the following corollary.

Corollary 1. Necessary and sufficient condition for the infinitesimal bending of the spherical curve C, determined by the field $\mathbf{z}$ given by Eqs. (7), to be on the sphere S with a given precision is that the field $\mathbf{z}$ is of constant intensity.

Proof. Since for the field $\mathbf{z}$ holds $\dot{\mathbf{z}} = \frac{1}{a}\mathbf{r}$, the condition (8) reduces to $\dot{\mathbf{z}} \cdot \mathbf{z} = 0 \leftrightarrow (\mathbf{z} \cdot \mathbf{z})^{\cdot} = 0 \leftrightarrow \left|\left|\mathbf{z}\right|\right| = const.$ ■

4. VIVIANI'S CURVE

Viviani's curve is obtained as an intersection of a sphere with a cylinder that is tangent to the sphere and passes through two poles of the sphere. The parametric representation is

$$\mathbf{r}(t) = (a(1+\cos t), a\sin t, 2a\sin\frac{t}{2}), \quad t \in [-2\pi, 2\pi], \quad a > 0.$$

It is easy to see that the vector field

$$\mathbf{z}(t) = (\sin t, -\cos t, 0)$$

is an infinitesimal bending field for which all bent curves are closed, i.e. $\mathbf{z}(-2\pi) = \mathbf{z}(2\pi)$. In Fig.2 we can see deformed Viviani's curve for $\epsilon = 0.5$ and $\epsilon = 1$. The red color denotes original curve, and blue deformed ones.

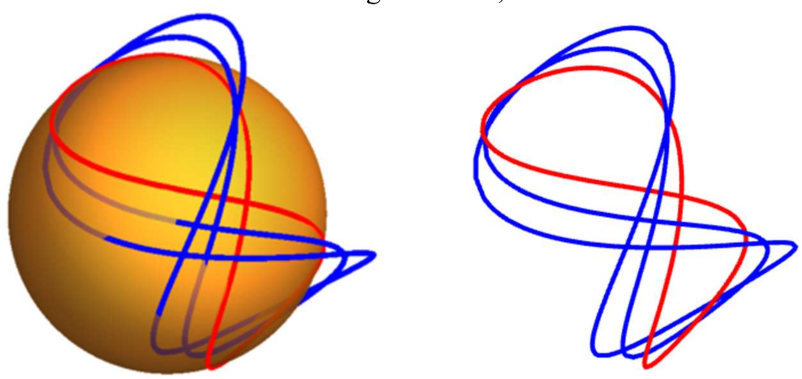

Figure 2: Viviani's curve and its infinitesimal bending for $\epsilon = 0.5$ and $\epsilon = 1$.

Let us examine whether this infinitesimal bending lies with a given precision on the sphere

$$S: ||\mathbf{r}||^2 = 4a^2$$

containing the initial Viviani's curve. Since

$$\mathbf{r}_\epsilon = (a(1+\cos t) + \epsilon\sin t, a\sin t - \epsilon\cos t, 2a\sin\frac{t}{2})$$

we easy obtain

$$||\mathbf{r}_\epsilon||^2 = 4a^2 + 2a\epsilon\sin t + \epsilon^2.$$

We conclude that bent curves are not approximately on the sphere S. However, the points for $t = k\pi \leftrightarrow \sin t = 0$ are on the sphere S with a given precision.

5. SPHERICAL CURVES WITH CONSTANT SLOPE

Spherical curve with constant slope (spherical helix) has the following parametric equation

$$\boldsymbol{r}(t)=\begin{cases}(a+b)\cos t-b\cos\frac{a+b}{b}t\\(a+b)\sin t-b\sin\frac{a+b}{b}t\\2\sqrt{ab+b^2}\cos\frac{a}{2b}t\end{cases}$$

$a\geq b>0$. It lies on the sphere of the radius $a+2b$. The vector field

$$\mathbf{z}(t)=(\frac{2b}{a+2b}\cos\frac{a+2b}{2b}t,\frac{2b}{a+2b}\sin\frac{a+2b}{2b}t,0)$$

is an infinitesimal bending field. In Fig.3 we have deformed spherical helix $(a=1,b=1)$ for $\epsilon=0.5$ and $\epsilon=1$ together with original curve (red colour).

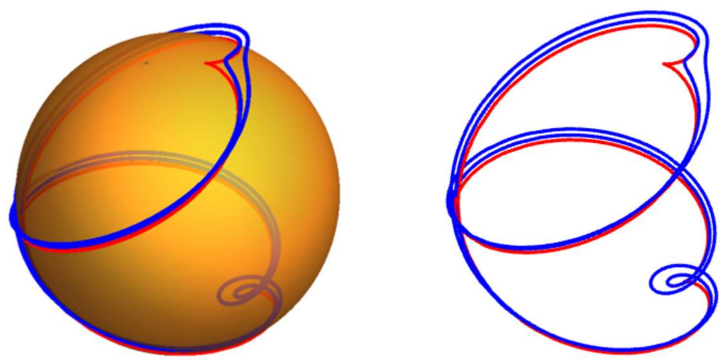

Figure 3: Spherical helix (1,1) (spherical cardioid) and its infinitesimal bending for $\epsilon=0.5$ and $\epsilon=1$.

By checking the condition

$$||\mathbf{r}_\epsilon||^2=(a+2b)^2+o(\epsilon)$$

we conclude that this infinitesimal bending is not on the initial sphere. More precisely, it holds

$$||\mathbf{r}_\epsilon||^2=(a+2b)^2+\epsilon\frac{4ab}{a+2b}\cos\frac{a}{2b}t+\epsilon^2\frac{4b^2}{(a+2b)^2}.$$

The points for $\cos\frac{a}{2b}t=0$ are on the initial sphere with a given precision.

6. CONCLUSIONS

In this paper we point out to the possibility of infinitesimal bending of spherical curves. We give explicit formulas for bending fields with appropriate graphic illustration using program packet Mathematica.

Many geometric magnitudes of curves are changed during the process of infinitesimal bending. For instance, in Fig.4 we can see how the curvature and the torsion of the Viviani's curve is changed for different values of the infinitesimal ϵ. Our next step is investigation of these changes and finding the appropriate variations.

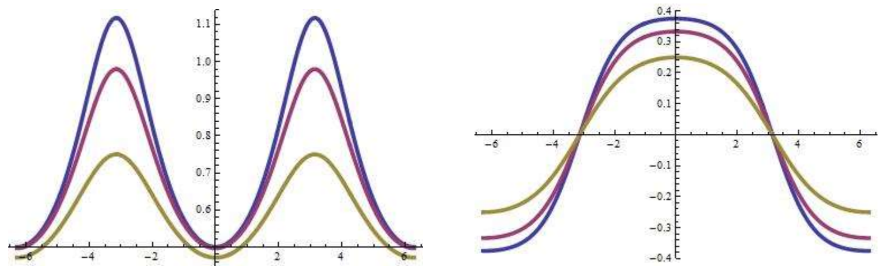

Figure 4: Curvature and torsion of Viviani's curve under infinitesimal bending for $\epsilon = 0$, $\epsilon = 0.5$ and $\epsilon = 1$.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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References

[1] V. A. Alexandrov, *On the total mean curvature of non-rigid surfaces*, Sibirsk. Mat. Zh. 50 (5) (2009) 963–996.
[2] N. V. Efimov, *Kachestvennye voprosy teorii deformacii poverhnostei,* UMN 3.2, (1948), 47-158.
[3] Y. Funakoshi, M. Hashizume, N. Ito, T. Kobayashi and H. Murai, *A distance on the equivalence classes of spherical curves generated by deformations of type RI*, Journal of Knot Theory and Its Ramifications Vol. 27, No. 12, 1850066 (2018).
[4] M. Hashizume, N. Ito, *New deformations on spherical curves and Östlund conjecture*, Topology and its Applications 301, 2021, 107508.
[5] I. Ivanova-Karatopraklieva, I. Sabitov, *Bending of surfaces II*, J. Math. Sci. (N. Y.) 74 (3) (1995), 997–1043.

[6] T. Kobayashi and S. Kobayashi, *Stable double point numbers of pairs of spherical curves*, JP Journal of Geometry and Topology Vol. 22, No. 2, (2019), 129 – 163.
[7] M. S. Najdanović, *Infinitesimal bending influence on the Willmore energy of curves*, Filomat 29(10) (2015) 2411–2419.
[8] M. S Najdanović, Lj. S. Velimirović, S. R. Rančić, *The total torsion of knots under second order infinitesimal bending*, Applicable Analysis and Discrete Mathematics 15 (2) (2021), 283–294.
[9] M. Najdanović, M. Maksimović, Lj. Velimirović, *Curves on ruled surfaces under infinitesimal bending*, Bulletin of Natural Sciences Research, Vol.11, No.1, 2021, pp. 38-43.
[10] S. Rancić, M. Najdanović, Lj. Velimirović, *Total normalcy of knots*, Filomat 33:4 (2019), 1259–1266.
[11] I. Vekua, *Obobschennye analiticheskie funkcii,* Moskva, 1959.
[12] Lj. Velimirović, *Change of geometric magnitudes under infinitesimal bending*, Facta Universitatis 3 (11), (2001), 135-148.
[13] Lj. Velimirović, *Infinitesimal bending*, Faculty of Science and Mathematics, Nis, (2009) ISBN 86-83481-42-5.
[14] Lj. S. Velimirović, M. S. Ćirić, M. Lj. Zlatanović, *Bendings of spherical curves*, 25th national and 2st international scientific conference moNGeometrija 2010, 657—667.

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