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CHAIN CONNECTED SET IN A SPACE

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Abstract. The paper gives a generalization of connectedness and chain connectedness of a space that is more general than a topological space and it consists of a set and a family of coverings of the set. In these spaces we define the notions of connected and chain connected sets, that are generalization of the notions with the same name in a topological spaces ([1]), and we study their properties. Also, the notions of a chain relation and a chain component in a space that are generalization of a chain relation and a chain component in a topological space ([1]), are defined and their properties are presented. Some new results for topological spaces are also provided.

1. Introduction

The definitions of connectedness by using the notion of chain as well as chain connectedness in a topological space, are given and theirs properties are studied in [1]-[5].

In this paper we use the notions and properties from article [1] and we generalize the notions to a space that is more general than a topological space and it consists of a set and a family of coverings of the set.

2. Space, subspace and chain

A space is a set X with added structure. By a space in this paper we understand the notion given in the next definition. By a covering of X we understand a covering of X in X.

Definition 2.1. ***The space*** *$X=(X,\underline{\mathcal{U}})$ is a set X together with a family of coverings $\underline{\mathcal{U}}=\{\mathcal{U}_\alpha \mid \alpha\in I\}$ of X.*

In this paper by a covering $\mathcal{U}$ of X we understand a covering that is an element of the family of coverings of X i.e. $\mathcal{U}\in\underline{\mathcal{U}}$.

Definition 2.2. ***Subspace*** *Y of the space $X=(X,\underline{\mathcal{U}})$, is the set Y with the family of coverings $\underline{\mathcal{U}_Y}=\underline{\mathcal{U}}\cap Y=\{\mathcal{U}_\alpha\cap Y \mid \alpha\in I\}$.*

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In this paper by subset $Y \subseteq X$ we understand the subspace Y of the space X.

Definition 2.3. *Let $\mathcal{U}$ be a covering of the set X and $x, y \in X$. **A chain in $\mathcal{U}$ that connects x and y** (from x to y, from y to x) is a finite sequence of sets $U_1, U_2, ..., U_n$ of $\mathcal{U}$ such that $x \in U_1$, $y \in U_n$ and $U_i \cap U_{i+1} \neq \varnothing$ for every $i = 1, 2, ..., n-1$.*

If X is a topological space, then the topology of X generate a family $\underline{\mathcal{U}}$, of all coverings $\mathcal{U}$ of X that consists of open sets. So, every topological space can be considered as a space that consists of a set X and a family $\underline{\mathcal{U}}$. When a topological space is considered as a space, this space is meant. Topological subspace Y of X is a subspace of the space X.

3. Chain connected set in a space

Using the notion of a chain we define the notion of chain connected set in a space.

Let X be a space and $C \subseteq X$.

Definition 3.1. *The set C is **chain connected** in X, if for every covering $\mathcal{U}$ of X and every $x, y \in C$, there exists a chain in $\mathcal{U}$ that connects x and y.*

Let X be a space and $C \subseteq Y \subseteq X$.

The first property of a chain connected set, shown in the next theorem, is an implication of chain connectedness from a space to each of its super spaces (X is a super space of C, if C is a subspace of X).

Theorem 3.2. *If C is chain connected in Y, then C is chain connected in X.*

Proof. Let C be chain connected in Y and $\mathcal{U}$ be a covering of X. Then:

$$\mathcal{U}_Y = \mathcal{U} \cap Y = \{U \cap Y | U \in \mathcal{U}\}$$

is a covering of Y. Since C is chain connected in Y, it follows that for every two points $x, y \in X$, there exists a chain $U_1 \cap Y, U_2 \cap Y, ..., U_n \cap Y$ of elements of $\mathcal{U}_Y$. Then since $U_i \cap U_{i+1} \neq \varnothing$ for every $i = 1, 2, ..., n-1$ and

$U_i \in \mathcal{U}$ for every $i = 1,2,...,n$, the sequence $U_1, U_2, ..., U_n$ is a chain in $\mathcal{U}$ that connects x and y. It follows that C is chain connected in X.■

Remark 3.3. *The most important case of the previous theorem is when* $Y = C$.■

Example 3.4. Consider the space $X = \{1,2,3\}$ with the family with one covering:

$$\mathcal{U} = \{\{1,2\},\{2,3\}\}.$$

The set $Y = \{1,3\}$ is chain connected in X, but it is not chain connected in Y since there does not exist a chain in $\mathcal{U}_Y = \mathcal{U} \cap Y = \{\{1\},\{3\}\}$ that connects 1 and 3.■

The next claim, which directly follows from the definition, shows that each subset of a chain connected set in a space is chain connected in the same space.

Remark 3.5. *If the set C is chain connected in X, then each subset of C is chain connected in X.*■

We will give criteria for chain connectedness in a space, using the notion of infinite star of a covering [1].

Let X be a space and $C \subseteq X$.

Let $\mathcal{U}$ be a covering of X and $x \in X$. Then the set $st(x,\mathcal{U})$ is a union of all $U \in \mathcal{U}$ which have nonempty intersection with x. The set:

$$st^n(x,\mathcal{U}) = st\left(st^{n-1}(x,\mathcal{U})\right) \text{ and } st^\infty(x,\mathcal{U}) = \bigcup_{n=1}^{\infty} st^n(x,\mathcal{U}).$$

Theorem 3.6. *The set C is chain connected in X, if and only if for every $x \in C$ and every covering $\mathcal{U}$ of X, $C \subseteq st^\infty(x,\mathcal{U})$.*■

Corollary 3.7. *The space X is chain connected in X, if and only if for every $x \in X$ and every covering $\mathcal{U}$ of X, $X = st^\infty(x,\mathcal{U})$.*■

4. Chain relation and chain components

Let $X = (X, \underline{\mathcal{U}})$ be a space and $x \in C \subseteq X$.

Definition 4.1. *The **chain component** of the point x of C in X, denoted by $V_{CX}(x)$, is the maximal chain connected subset of C in X that contains x.*■

Proposition 4.2. *The set $V_{CX}(x)$ consists of all elements $y \in C$, such that for every covering $\mathcal{U} \in \underline{\mathcal{U}}$ there exists a chain in $\mathcal{U}$ that connects x and y.*

If $C = X$, then we use notation $V_X(x)$ or $V(x)$ if we work only with the space X, for $V_{XX}(x)$. Clearly $V_{CX}(x) = C \cap V_X(x)$.

Example 4.3. For the space $X = \{1,2,3\}$ with the family consisting of one covering $\mathcal{U} = \{\{1,2\},\{2,3\}\}$, and the set $C = \{1,3\}$:

$$V_X(1) = \{1,2\} \text{ and } V_{CX}(1) = \{1\}.$$

The set C is chain connected in X if C is subset of $V_X(x)$ for every $x \in C$.

We denote by $U_X(C)$ or $U(C)$, the set that consists of all elements $y \in X$, such that for every covering $\mathcal{U} \in \underline{\mathcal{U}}$ there exists a chain in $\mathcal{U}$ that connects some $x \in C$ and y. This set is a union of chain components.

If C is a chain connected set in X, since for every $x, y \in C$ the chain components $V(x)$ and $V(y)$ coincide i.e. $V(x) = V(y)$, it follows that $U(C)$ is chain component and it is denoted by $V(C)$. Clearly $C \subseteq V_X(C)$ and $V(C) = V(x)$ for every $x \in C$.■

Remark 4.4. *If the set C is chain connected in X, then each subset of $V(C)$ is chain connected in X.*■

When X is a topological space, then it is obvious that the next statement improves the remark 2.2 from [1].

Remark 4.5. *If the set C is chain connected in a topological space X, then each subset of $V(C)$ is chain connected in X.*■

Let X be a space and $x, y \in X$.

Definition 4.6. *Th element x is* ***chain related*** *to y in X, and we denote it by $x \sim y$ if for every covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects x and y.*

The chain relation in a space is an equivalence relation and it depends on the set X and the family of coverings of X.

Remark 4.7. *The set C is chain connected in X if and only if for every $x, y \in C$, $x \sim y$.* ■

Therefore C is not chain connected in X if and only if there exist $x, y \in C$ such that $x \nsim y$.

The chain relation decomposes the space into classes. The classes are chain components.

Let $x, y \in C$. If $y \in V_{CX}(x)$, then $V_{CX}(x) = V_{CX}(y)$. If $V_{CX}(x) \neq V_{CX}(y)$, then $V_{CX}(x) \cap V_{CX}(y) = \emptyset$. As a consequence, the next proposition is valid.

Proposition 4.8. *For every $x \in C$, $V_{CX}(x) = C \cap V_{XX}(x)$. Each chain component of X in X contains at most one chain component of C in X.* ■

Proposition 4.9. *For every $x \in C$,*

$$V_{CC}(x) \subseteq V_{CX}(x) = \bigcup_{y \in V_{CX}(x)} V_{CC}(y) \subseteq V_{XX}(x). \blacksquare$$

The proposition shows that every chain component of C in X is a union of chain components of C in C and is a subset of chain component of X in X.

Proposition 4.10. *The set of all chain connected subsets of C in X consist of all chain components and their subsets.* ■

5. Properties of chain connected sets that consist chain components

Next we turn to a union of chain connected sets in a topological space.

Lemma 5.1. *Let $C, D \subseteq X$. If C and D are chain connected in X and $V(C) \cap V(D) \neq \emptyset$, where $V(C)$ and $V(D)$ are chain components of C and D respectively, then the union $V(C) \cup V(D)$ is chain connected in X and*

$$V(C) \cup V(D) = V(C) = V(D).$$

Proof. Let $\mathcal{U}$ be a covering of X and $x, y \in V(C) \cup V(D)$. If $x, y \in V(C)$ or $x, y \in V(D)$, then since $V(C)$ and $V(D)$ are chain connected, there exists a chain in $\mathcal{U}$ that connects x and y. If $x \in V(C)$ and $y \in V(D)$, it follows that firstly there exists $z \in V(C) \cap V(D)$, and secondly that there exist chains in $\mathcal{U}$ that connect x with z, and z with y, from which it follows that there is a chain in $\mathcal{U}$ that connects x and y. So $V(C) \cup V(D)$ is chain connected in X.

Since $V(C) = V(x)$ is chain component of some $x \in C$, and $V(C) \cup V(D)$ is a chain connected set that contain x it follows that $V(C) \cup V(D) = V(C)$. Similarly $V(C) \cup V(D) = V(D)$.■

Corollary 5.2. *Let $C, D \subseteq X$. If C and D are chain connected in X and $V(C) \cap V(D) \neq \emptyset$, where $V(C)$ and $V(D)$ are chain components of C and D respectively, then the union $C \cup D$ is chain connected in X.*

Theorem 5.3. *Let $C_i, i \in I$ be a family of chain connected subspaces of X. If there exists $i_0 \in I$ such that for every $i \in I$, $V(C_{i_0}) \cap V(C_i) \neq \emptyset$, then the union $\bigcup_{i \in I} V(C_i)$ is chain connected in X* and *$\bigcup_{i \in I} V(C_i) = V(C_i)$ for every $i \in I$,*

Proof. Let $\mathcal{U}$ be a covering of X and $C_i, i \in I$ be a family of chain connected subspaces of X. Let $x, y \in \bigcup_{i \in I} V(C_i)$, i.e. $x \in V(C_x)$ and $y \in V(C_y)$ for some $x, y \in I$.

Since $V(C_{i_0}) \cap V(C_i) \neq \emptyset$ for every $i \in I$, from the previous lemma, it follows that $V(C_{i_0}) \cup V(C_x)$ is chain connected in X. Similarly $V(C_{i_0}) \cup V(C_y)$ is chain connected in X. Then because $C_{i_0} \neq \emptyset$, from the previous lemma it follows that $V(C_{i_0}) \cup V(C_x) \cup V(C_y)$ is chain connected in X, i.e. for every covering $\mathcal{U}$ of X, there exists a chain in $\mathcal{U}$ that connects x and y. So $\bigcup_{i \in I} V(C_i)$ is chain connected in X.

Since $V(C_i) = V(x_i)$ is chain component of some $x_i \in C_i$ for every $i \in I$, and $\bigcup_{i \in I} V(C_i)$ is a chain connected set that contain x it follows that:

$$\bigcup_{i \in I} V(C_i) = V(C_i) \text{ for every } i \in I. \blacksquare$$

Corollary 5.4. *Let $C_i, i \in I$ be a family of chain connected subspaces of X. If there exists $i_0 \in I$ such that for every $i \in I$, $C_{i_0} \cap C_i \neq \emptyset$, then the union $\bigcup_{i \in I} C_i$ is chain connected in X.*

If X is a topological space then the next two statements which directly follow from the lemma 5.1 and the theorem 5.3, respectively, improves the lemma 3.1 and theorem 3.6 from [1].

Corollary 5.5. *If C and D are chain connected sets in a topological space X and $V(C) \cap V(D) \neq \emptyset$, where $V(C)$ and $V(D)$ are chain components of C and D respectively, then the union $V(C) \cup V(D)$ is chain connected in X.*

Theorem 5.6. *Let $C_i, i \in I$ be a family of chain connected subspaces of a topological space X. If there exists $i_0 \in I$ such that for every $i \in I$, $V(C_{i_0}) \cap V(C_i) \neq \emptyset$, then the union $\bigcup_{i \in I} V(C_i)$ is chain connected in X.* $\blacksquare$

Corollary 5.7. *If every two points x and y of $C \subseteq X$ are in a chain connected set C_{xy} in X, then C is chain connected in X.* $\blacksquare$

6. CONCLUSIONS

The notion of connectedness by using the standard definition cannot be generalized from a topological space to a more general space, without generalizing the topology of the space, since it is related to it. However connectedness defined by chain ([1]), as well as its generalization chain connectedness to a pair of a topological space and its subspace, can, such that instead of families of all coverings of open sets, subfamilies of coverings of arbitrary sets will be considered. The generalizations can also be defined on even more general structures, such as a set of subsets of a given set, i.e. to define connectedness in this set.

The paper gives a generalization of connectedness and chain connectedness of a space that is more general than a topological space and it consists of a set and a family of coverings of the set. In these spaces, the notion of chain connected set, as well as the notion of chain relation are defined and their properties are presented. A number of statements from [1] cannot be generalized to the space level. Two examples for spaces that are not topological, are given, to be shown that one statement must not be true in the converse direction and in one statement two sets are not equal. Also, the special cases of claims 5.1 and 5.3 of the paper provide a new results at a level of topological spaces. They reduce to stronger claims than the corresponding claims lemma 3.1 and theorem 3.6 from [1], since they are expressed by using chain components instead of closed sets.

The generalizations to a space can be done to other topological notions and spaces.

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