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Cantor's Intersection Theorem in $(3, 1, \nabla)$-G-metrizable spaces

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Abstract
In this article we prove some properties of $(3, 1, \nabla)$-G-metrizable spaces and establish analogous Cantor's intersection theorem for those kinds of spaces.

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1 Introduction

In metric spaces the Cantor's intersection theorem is well known fact and characterizes the metric completeness. It states that for an arbitrary complete metric space X, a sequence of nonempty, nested and closed subsets of X whose diameters tend to 0, has a single point intersection, and vice versa. Different type of generalisations have been obtained considering generalization of the spaces or the notion of decreasing sets.

Generalizations of metric spaces have been considered in lot of papers by many authors: Menger [14], Aleksandrov, Nemytskii [1], Mamuzić [15], Gähler [12], Nedev, Choban [18, 19, 20], Kopperman [13], Dhage, Mustafa, Sims [5, 16]. The notion of an (n, m, ρ)-metric, $n > m$, generalizing the usual notion of a pseudometric (the case $n = 2, m = 1$), and the notion of an $(n + 1)$-metric (as in [14] and [12]) was introduced in [6]. Connections between some of the topologies induced by a $(3, 1, \rho)$-metric and topologies induced by a pseudo-o-metric, o-metric and symmetric (as in [19]), are given in [7]. Some other characterizations of $(3, j, \rho)$-metrizable topological spaces, $j \in \{1, 2\}$, are given in [3, 4, 8, 9].

In this article we will consider only $(3, 1, \nabla)$-metrics, i.e. $(3, 1, \rho)$-metrics where $\rho = \nabla = \{(x, x, y)|x, y \in M\}$. The concept of subbasis that forms a topological space has also been considered by Gähler in [12]. Similarly we define the topology $\tau(G, d)$ for a $(3, 1, \rho)$-metric d generated by the subbasis of all ϵ-balls with center at (x, y), defined as in [3]. Here we gained some properties of $(3, 1, \nabla)$-G-metrizable spaces which combined with some assumptions enabled us to prove a variant of Cantor's intersection theorem in these kinds of spaces.

2 Preliminaries

We use basic definitions for $(3,1,\rho)$-metric spaces and $(3,1)$-metric spaces, as in [3].
Let $M \neq \varnothing$ and $M^{(3)} = M^3/\alpha$, where α is the equivalence relation on M^3 defined by:

$$(x,y,z)\alpha(u,v,w) \Leftrightarrow \pi(u,v,w) = (x,y,z),$$

where π is a permutation.

Note 2.1. *We will use the same notation (x,y,z) for the class in $M^{(3)}$ containing the triplet (x,y,z).*

Let ρ be a subset of $M^{(3)}$. We consider the following conditions for such set.

$(E0)$ $(x,x,x) \in \rho$, for any $x \in M$; and

$(E1)$ $(a,y,z),(x,a,z),(x,y,a) \in \rho \implies (x,y,z) \in \rho$, for any $x,y,z,a, \in M$.

Definition 2.2. *If ρ satisfies $(E0)$ and $(E1)$ we say that ρ is a $(3,1)$-equivalence.*

We give the following trivial examples.

Example 2.3. *The set $\Delta = \{(x,x,x)|x \in M\}$ is a $(3,1)$-equivalence on M.*

Example 2.4. *The set $\nabla = \{(x,x,y)|x,y \in M\}$ is a $(3,1)$-equivalence on M.*

Let $d : M^{(3)} \to \mathbb{R}_0^+$. We consider the following conditions for such a map.

$(M0)$ $d(x,x,x) = 0$, for any $x \in M$; and

$(M1)$ $d(x,y,z) \leq d(a,x,z) + d(x,a,z) + d(x,y,a)$, for any $x,y,z,a \in M$.

Lemma 2.5. *Let $d : M^{(3)} \to \mathbb{R}_0^+$ and $\rho_d = \{(x,y,z) \in M^{(3)}|d(x,y,z) = 0\}$. If d satisfies $(M0)$ and $(M1)$, then ρ_d is a $(3,1)$-equivalence.*

Proof. It follows directly from the previous definition. □

Definition 2.6. *Let $d : M^{(3)} \to \mathbb{R}_0^+$ and $\rho = \rho_d$.*
i) If d satisfies $(M0)$ and $(M1)$ we say that d is a $(3,1,\rho)$-metric on M, and the pair (M,d) is said to be a $(3,1,\rho)$-metric space.
ii) If d is a $(3,1,\Delta)$-metric on M, we say that d is a $(3,1)$-metric on M, and the pair (M,d) is said to be a $(3,1)$-metric space.

Again we state certain examples.

Example 2.7. *Let M be a nonempty set. The map $d : M^{(3)} \to \mathbb{R}_0^+$ defined by:*

$$d(x,y,z) = \begin{cases} 0 & ,(x,y,z) \in \Delta \\ 1 & , \textit{otherwise} \end{cases}$$

is a $(3,1)$-metric on M (the discrete 3-metric).

Proof. Follows directly from Definition 2.6. □

Example 2.8. *Let $D : M^2 \to \mathbb{R}_0^+$ be a metric on M. The map $d : M^{(3)} \to \mathbb{R}_0^+$ defined by:*

$$d(x, y, z) = \frac{D(x, y) + D(x, z) + D(y, z)}{2},$$

is a $(3, 1)$-metric on M.

Proof. Follows directly from Definition 2.6. □

2.1 Topological framework

In this subsection we define a suitable framework in which we are considering our results. We define all the necessary topological notions in the sequel.

Definition 2.9. *Let (M, d) be a $(3, 1, \rho)$ metric space and $A \subseteq M, A \neq \emptyset$. We say that A is bounded if there is an $M > 0$ such that $d(x, y, z) \leq M$, for all $x, y, z \in M$.*
If A is bounded, we define the diameter of A as

$$\mathrm{diam}A = \sup\{d(x, y, z) | x, y, z \in M\}.$$

If A is not bounded, we write $\mathrm{diam}A = \infty$.

Definition 2.10. *We say that a sequence $(x_n)_{n=1}^{\infty}$ in a $(3, 1, \rho)$-metric space (M, d) is G-convergent if there is an $x \in M$ such that $d(x_n, x, y) \to 0$ as $n \to \infty$ for each $y \in M$.*

For simplicity we use the notation $x_n \to x$ as $n \to \infty$ for G-convergence of the sequence $(x_n)_{n=1}^{\infty}$.

Definition 2.11. *We say that a sequence $(x_n)_{n=1}^{\infty}$ in a $(3, 1, \rho)$-metric space (M, d) is G-Cauchy if $d(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.*

Definition 2.12. *We say that a $(3, 1, \rho)$-metric space (M, d) is G-complete if each G-Cauchy sequence is G-convergent (with respect to d).*

Definition 2.13. *Let d be a $(3, 1, \rho)$-metric on M, $x, y \in M$ and $\epsilon > 0$. We define the ϵ-ball with center at (x, y) and radius ϵ to be the set*

$$B(x, y, \epsilon) = \{z | z \in M, d(x, y, z) < \epsilon\}.$$

Definition 2.14. *For a $(3, 1, \rho)$-metric d on M, we define the topology $\tau(G, d)$ on M to be the topology generated by all ϵ-balls $B(x, y, \epsilon)$ for all $x, y \in M$, i.e. the topology whose base is the set of all finite intersections of all ϵ-balls $B(x, y, \epsilon)$ for all $x, y \in M$.*

Definition 2.15. *We say that a topological space (M, τ) is $(3, 1, \rho)$-G-metrizable if there is a $(3, 1, \rho)$-metric d on M such that $\tau = \tau(G, d)$.*

Definition 2.16. *Let M be a set and $\mathcal{M} = \mathcal{P}(M)$ be the power set of M. We say that a sequence $(F_n)_{n=1}^{\infty}$ of subsets of M is decreasing (in $\mathcal{M}$) if $F_{n+1} \subseteq F_n$, for each $n \in \mathbb{N}$.*

Lemma 2.17. *Let (M, d) be a $(3, 1, \rho)$-metric space and $x, y \in M$ are fixed. If $z_n \to z$ as $n \to \infty$, then $d(z_n, x, y) \to d(z, x, y)$ as $n \to \infty$.*

Proof. Let $z_n \to z$ as $n \to \infty$ and $x, y \in M$. Taking $(M1)$ in to account one obtains

$$d(z_n, x, y) \leq d(z, x, y) + d(z_n, z, y) + d(z_n, x, z),$$

i.e.

$$d(z_n, x, y) - d(z, x, y) \leq d(z_n, z, y) + d(z_n, x, z), \tag{1}$$

for arbitrary $n \in \mathbb{N}$. Interchanging z_n with z implies

$$d(z, x, y) - d(z_n, x, y) \leq d(z, z_n, y) + d(z, x, z_n). \tag{2}$$

Combined inequalities (1) and (2) impliy

$$|d(z_n, x, y) - d(z, x, y)| \leq d(z_n, z, y) + d(z_n, x, z).$$

The convergence of the sequence $(z_n)_{n=1}^{\infty}$ implies that the right-hand side tends to 0 when $n \to \infty$. Therefore, the proof is completed. □

Lemma 2.18. [4] Let (M, τ) be a $(3, 1, \nabla)$-metrizable space, via $(3, 1, \nabla)$-metric d. A subset U from M is open iff for any $x \in U$ there are finite number of points $a_1, a_2, ..., a_n \in M$ and $\epsilon_1, \epsilon_2, ..., \epsilon_n > 0$ such that $x \in \cap_{n=1}^{\infty} B(x, a_i, \epsilon_i) \subseteq U$.

Lemma 2.19. *Let (M, τ) be a $(3, 1, \nabla)$-metrizable space, via $(3, 1, \nabla)$-metric d. A sequence $(x_n)_{n=1}^{\infty}$ G-converges to $x \in M$ iff for any $U \in \tau$ such that $x \in U$, there exists an $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$.*

Proof. Follows directly from the previous lemma. □

3 Main results

We impose additional conditions on the set M.
Let (M, τ) be a $(3, 1, \nabla)$-metrizable space, via $(3, 1, \nabla)$-metric d, satisfying the conditions:

(1) For $x, y \in M, x \neq y$ there is a sequence $(z_n)_{n=1}^{\infty}$ in $M \setminus \{x, y\}$ and $z \in M \setminus \{x, y\}$ such that $z_n \to z$ and $d(z_n, x, y) \to 0$ as $n \to \infty$,

(2) If there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ from G-Cauchy sequence $(x_n)_{n=1}^{\infty}$ such that $x_{n_k} \to x$ as $k \to \infty$, then $x_n \to x$ as $n \to \infty$.

Our first concern is the existence of nontrivial example of such space (M, τ). The condition (1) seems really restrictive. Nerveless, such spaces exist. Our example of this type of space is geometrically motivated, so the details are left for an interested reader.

Example 3.1. *Let M be the set consisted of all triples (x, y, z) of not collinear points in $\mathbb{R}^2$ and all the triples (x, y, z) of points in $\mathbb{R}^2$ such that at least two of them are the same. Define $d(x, y, z)$ by the area of the triangle in $\mathbb{R}^2$ with vertices x, y, z. It is not difficult to confirm that $(M0), (M1)$, together with (1) and (2) are satisfied.*

Lemma 3.2. *Let $A \subseteq M$. Then* $\mathrm{diam} A = \mathrm{diam} \overline{A}$.

Proof. It is obvious that $\mathrm{diam} A \leq \mathrm{diam} \overline{A}$.
Let $x, y, z \in \overline{A}$. We consider the following cases.

1^0 If $x, y, z \in A$, then $d(x, y, z) \leq \mathrm{diam} A$.

2^0 If $x \in \overline{A} \setminus A$ and $y, z \in A$, then for each $\epsilon > 0$ there exists $u \in A \cap B(x, y, \epsilon) \cap B(x, z, \epsilon)$. Then

$$\begin{aligned} d(x, y, z) &\leq d(u, y, z) + d(x, u, z) + d(x, y, u) \\ &< 2\epsilon + \mathrm{diam} A. \end{aligned}$$

3^0 If $x, y \in \overline{A} \setminus A$ and $z \in A$, then for each $\epsilon > 0$ there exist u, v such that $u \in A \cap B(x, y, \epsilon) \cap B(x, z, \epsilon)$ and $v \in A \cap B(y, u, \epsilon) \cap B(y, z, \epsilon)$. Then

$$\begin{aligned} d(x, y, z) &\leq d(u, y, z) + d(x, u, z) + d(x, y, u) \\ &< d(u, y, z) + 2\epsilon \\ &\leq d(v, y, z) + d(u, v, z) + d(u, y, v) + 2\epsilon \\ &< 4\epsilon + \mathrm{diam} A. \end{aligned}$$

4^0 If $x, y, z \in \overline{A} \setminus A$, then for each $\epsilon > 0$ there exist u, v, t such that $u \in A \cap B(x, y, \epsilon) \cap B(x, z, \epsilon)$, $v \in A \cap B(y, u, \epsilon) \cap B(y, z, \epsilon)$ and $t \in A \cap B(z, u, \epsilon) \cap B(z, v, \epsilon)$.Then

$$\begin{aligned} d(x, y, z) &\leq d(u, y, z) + d(x, u, z) + d(x, y, u) \\ &< d(u, y, z) + 2\epsilon \\ &\leq d(v, y, z) + d(u, v, z) + d(u, y, v) + 2\epsilon \\ &< 4\epsilon + d(u, v, z) \\ &\leq 4\epsilon + d(t, v, z) + d(u, t, z) + d(u, v, t) \\ &< 6\epsilon + \mathrm{diam} A. \end{aligned}$$

Regardless of the cases, arbitrariness of ϵ implies the claim. □

Next will prove analogous theorem of Cantor's intersection theorem in $(3, 1, \nabla)$-G-metrizable spaces.
Let us consider another condition.

(C) For each decreasing sequence $(F_n)_{n=1}^{\infty}$ of closed subsets of M such that $\lim_{n\to\infty} \mathrm{diam} F_n = 0$ the set $\cap_{n=1}^{\infty} F_n$ consists of a single point.

Theorem 3.3. *If (M,τ) is G-complete, then the condition (C) is satisfied.*

Proof. For each $n \in \mathbb{N}$, let $x_n \in F_n$. Since $(F_n)_{n=1}^{\infty}$ is a decreasing sequence, for $m, l \geq n$ we have $x_m, x_l \in F_n$ and

$$d(x_n, x_m, x_l) \leq \operatorname{diam} F_n \to 0,$$

as $n \to \infty$. Thus, the sequence $(x_n)_{n=1}^{\infty}$ is a G-Cauchy sequence. This means that there is an $x \in M$ such that $x_n \to x$ as $n \to \infty$. We will prove that $x \in \cap_{n=1}^{\infty} F_n$.
Let n be fixed arbitrary positive integer and $U \in \tau$ such that $x \in U$. Then there is a $k_0 \in \mathbb{N}$ such that $x_k \in U$ for all $k \geq k_0$. Thus, $x_k \in U \cap F_n$ for all $k \geq \max\{n, k_0\}$, i.e. $x \in \overline{F_n} = F_n$ (F_n is closed). So, $x \in \cap_{n=1}^{\infty} F_n$. Let us suppose that there is a $y \in M \setminus \{x\}$ such that $y \in \cap_{n=1}^{\infty} F_n$. From the condition (1) it follows that for each $\epsilon > 0$, there are a sequence $(z_n)_{n=1}^{\infty}$ in $M \setminus \{x, y\}$, $z \in M \setminus \{x, y\}$ such that $z_n \to z$, and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$0 \leq d(x, y, z_n) < \epsilon.$$

Letting $n \to \infty$ we obtain that $d(x, y, z) = 0$, which is a contradiction since $x \neq y \neq z \neq x$. □

Theorem 3.4. *If (M,τ) satisfies the condition (C), then (M,τ) is G-complete.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a G-Cauchy sequence in M and for each $n \in \mathbb{N}$ set $F_n = \{x_n, x_{n+1}, ...\}$. Then the sequence $(F_n)_{n=1}^{\infty}$ is decreasing and moreover, $(\overline{F_n})_{n=1}^{\infty}$ is decreasing sequence of closed sets. For $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$d(x_m, x_n, x_l) < \epsilon$$

for each $m, n, l \geq n_0$, and lemma 3.2 infers $\operatorname{diam}\overline{F_{n_0}} = \operatorname{diam} F_{n_0} \leq \epsilon$. But then $\operatorname{diam}\overline{F_n} \leq \epsilon$ for each $n \geq n_0$ meaning that $\lim_{n\to\infty} \operatorname{diam}\overline{F_n} = 0$. So, there is $x \in M$ such that $\cap_{n=1}^{\infty}\overline{F_n} = \{x\}$.
Let $z \in M$ be arbitrary. For each $n \in \mathbb{N}$ there is $y_n \in B(z, x, \frac{1}{n}) \cap F_n$. Thus, $(y_n)_{n=1}^{\infty}$ is a subsequence of $(x_n)_{n=1}^{\infty}$ such that $d(y_n, x, z) \to 0$ as $n \to \infty$, i.e. $y_n \to x$ as $n \to \infty$. From condition (2) it follows that $x_n \to x$ as $n \to \infty$, i.e. (M,τ) is G-complete. □

From the previous two theorems we obtain the analogous Cantor's intersection theorem in these kinds of spaces.

Corollary 3.5. *Let (M,τ) be a $(3,1,\nabla)$-metrizable space, via $(3,1,\nabla)$-metric d satisfying the conditions (1) and (2). Then (M,τ) is G-complete iff it satisfies the condition (C).*

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