Union of Mathematicians of Macedonia - ARMAGANKA

IX SEMINAR OF DIFFERENTIAL EQUATIONS AND ANALYSIS

and

1st CONGRESS OF DIFFERENTIAL EQUATIONS, MATHEMATICAL ANALYSIS AND APPLICATIONS

CODEMA 2020

Proceedings of the CODEMA 2020
Зборник на трудови од CODEMA 2020

Skopje, 2021

ISBN 978-608-4904-09-0 UDC: 512.554.3:512.643

Moore-Penrose Hermitian elements in rings with involution

Marina Tošić, Nataša Kontrec, Eugen Ljajko

Abstract

A Moore-Penrose invertible element in rings with involution is Moore-Penrose Hermitian if Moore-Penrose inverse is equal to the element itself. In this paper, we present a number of new characterizations of Moore-Penrose Hermitian elements in rings with involution in purely algebraic terms.

2000 *Mathematics Subject Classification*: 16B99, 16W10, 46L05
Key words: Moore-Penrose Hermitian elements, group inverse, Moore-Penrose inverse, ring with involution

1 Introduction

Let $\mathcal{R}$ be an associative ring with unity 1, and let $a \in \mathcal{R}$. Then a is group invertible if there is $a^\sharp \in \mathcal{R}$ such that

$$(1)\ aa^\sharp a = a, \quad (2)\ a^\sharp a a^\sharp = a^\sharp, \quad (3)\ aa^\sharp = a^\sharp a.$$

Recall that $a^\sharp$ is uniquely determined by previous equations. We use $\mathcal{R}^\sharp$ to denote the set of all group invertible elements of $\mathcal{R}$.

An involution $a \to a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(1)\ (a^*)^* = a, \quad (2)\ (a+b)^* = a^* + b^*, \quad (3)\ (ab)^* = b^* a^*.$$

In the rest of the paper, we assume that $\mathcal{R}$ is a ring with involution. An element $a \in \mathcal{R}$ satisfying $aa^* = a^*a$ is called normal.

The Moore-Penrose inverse of a, is the unique element $a^\dagger$ satisfying the equations

$$(1)\ aa^\dagger a = a, \quad (2)\ a^\dagger aa^\dagger = a^\dagger, \quad (3)\ aa^\dagger = (aa^\dagger)^*, \quad (4)\ a^\dagger a = (a^\dagger a)^*.$$

The subset of $\mathcal{R}$ consisting of elements of $\mathcal{R}$ that have a Moore-Penrose inverse will be denoted by $\mathcal{R}^\dagger$.

Several characterizations of elements $a \in \mathcal{R}^\dagger$ such that $aa^\dagger = a^\dagger a$ can be found in the literature (see [1], [6], [7], [8] and [10]). These elements are called EP. For a ring with involution $\mathcal{R}$, we will denote $\mathcal{R}^{EP} = \{a \in \mathcal{R}^\dagger : aa^\dagger = a^\dagger a\}$.

An element $a \in \mathcal{R}^\dagger$ satisfying $a^* a^\dagger = a^\dagger a^*$ is called star-dagger.

A characterization of nonnegative matrices which are equal to their Moore-Penrose inverse is derived by Berman in [2]. Many years ago, the concept of Moore-Penrose Hermitian elements in C^*-algebras was introduced by E. Boasso [3]: Let $\mathcal{A}$ be a C^*-algebra. A regular element $a \in \mathcal{A}$ is Moore-Penrose Hermitian if $a^\dagger = a$. The definition of Moore-Penrose Hermitian elements can be generalized to elements in rings with involution.

Definition 1.1 *If $\mathcal{R}$ is a ring with involution, and $a^\dagger$ is the Moore-Penrose inverse of $a \in \mathcal{R}^\dagger$, then the element a is called Moore-Penrose Hermitian if $a^\dagger = a$.*

The following result is well-know and frequently used in the rest of the paper.

Theorem 1.1 *[4, 9] For any $a \in \mathcal{R}^\dagger$, the following is satisfied:*

(i) $(a^\dagger)^\dagger = a$,

(ii) $a^* = a^\dagger aa^* = a^* aa^\dagger$.

In [2] and [5], authors used the representation of complex matrices to explore various property of Moore-Penrose Hermitian matrices. In this paper, we use a different approach, exploiting the structure of rings with involution to investigate Moore-Penrose Hermitian elements. We give several characterizations, and the proofs are based on ring theory only.

2 Results

E. Boasso proved the following result in [3].

Proposition 2.1 *Consider a C^*-algebra $\mathcal{A}$ and an element $a \in \mathcal{A}$. Then the following statements hold:*

(i) *Necessary and sufficient for a to be a Moore-Penrose Hermitian is $a = a^3$ and $(a^2)^* = a^2$,*

(ii) *If a is a Moore-Penrose Hermitian, then a^n also is, $n \in \mathcal{N}$.*

(iii) *The element a is a Moore-Penrose Hermitian if and only if a^* is.*

(iv) *If a is a Moore-Penrose Hermitian, then $\sigma(a) \subseteq \{0, -1, 1\}$, where $\sigma(a)$ denotes the spectrum of a.*

Observe that the previous proposition holds on Moore-Penrose Hermitian elements in rings with involution. Some new characterizations of Moore-Penrose Hermitian elements in rings with involution are given in the following results.

Theorem 2.1 *Let $\mathcal{R}$ be a ring with involution and $a \in \mathcal{R}^\dagger$. Then a is a Moore-Penrose Hermitian if and only if one of the following equivalent conditions holds:*

(i) *$a^\dagger$ is a Moore-Penrose Hermitian,*

(ii) *$a^* = (a^*)^3$ and $(a^2)^* = a^2$,*

(iii) *$a^\dagger = (a^\dagger)^3$ and $((a^\dagger)^2)^* = (a^\dagger)^2$,*

(iv) *$a = a^3$ and a is an EP element,*

(v) *$a^* = (a^*)^3$ and a^* is an EP element,*

(vi) *$a^\dagger = (a^\dagger)^3$ and $a^\dagger$ is an EP element,*

(vii) *$a = a^3$ and $a^\dagger aaa^\dagger = aa^\dagger a^\dagger a$,*

(viii) *$a^\dagger = (a^\dagger)^3$ and $a^\dagger aaa^\dagger = aa^\dagger a^\dagger a$,*

(ix) *$a^* = aaa^*$ (or $a^* = a^*aa$),*

(x) $a = a^*a^*a$ *(or* $a = aa^*a^*$*)*

(xi) $a = a^3$ *and* $a^\dagger = aa^*a^*$ *(or* $a = a^3$ *and* $a^\dagger = a^*a^*a$*),*

(xii) $a^\dagger a = aa$ *(or* $aa^\dagger = aa$*),*

(xiii) $aa^\dagger = a^\dagger a^\dagger$ *(or* $a^\dagger a = a^\dagger a^\dagger$*),*

(xiv) $a^\dagger = a^\dagger aa$ *(or* $a^\dagger = aaa^\dagger$*),*

(xv) $a = aa^\dagger a^\dagger$ *(or* $a = a^\dagger a^\dagger a$*).*

Proof. If a is is a Moore-Penrose Hermitian, then it commutes with its Moore-Penrose inverse and $a^\dagger = a$. It is not difficult to verify that conditions (iii)-(xiii) hold.

Conversely, to conclude that a is is a Moore-Penrose Hermitian, we show that the condition $a^\dagger = a$ is satisfied, or that the element is subject to one of the preceding already established conditions of this theorem.

(i) From $(a^\dagger)^\dagger = a^\dagger$ and $(a^\dagger)^\dagger = a$ follows $a^\dagger = a$. Hence, the element a is a Moore-Penrose Hermitian.

(ii) The conclusion follows by (i) of Proposition 2.1 and (iii) of Proposition 2.1.

(iii) The conclusion follows by (i) of Proposition 2.1 and (i).

(iv) Suppose that $a = a^3$ and $aa^\dagger = a^\dagger a$. Then

$$a^\dagger = (a^\dagger)^2 a = (a^\dagger)^2 a^3 = ((a^\dagger)^2 a)a^2 = a^\dagger a^2 = a.$$

(v) The conclusion follows by (iii) of Proposition 2.1 and (iv).

(vi) The conclusion follows by (i) and (iv).

(vii) From $a = a^3$ and $a^\dagger aaa^\dagger = aa^\dagger a^\dagger a$, we get

$$aa^\dagger = a^3 a^\dagger = aaa^\dagger aaa^\dagger = a^3 a^\dagger a^\dagger a = aa^\dagger a^\dagger a$$

and, similar,

$$a^\dagger a = a^\dagger a^3 = a^\dagger aaa^\dagger aa = aa^\dagger a^\dagger a^3 = aa^\dagger a^\dagger a.$$

Thus, $aa^\dagger = a^\dagger a$ and the condition (iv) is satisfied.

(viii) The conclusion follows by (i) and (vii).

(ix) If $a^* = aaa^*$, then

$$a^2 = a(a^*)^* = a(a^2a^*)^* = a^2(a^2)^* = (a^2a^*)a^* = a^*a^* = (a^2)^*.$$

Multiplying $a^2 = (a^2)^*$ by a, we obtain $a^3 = a(a^*)^2$. Now,

$$a = (a^*)^* = (a^2a^*)^* = a(a^*)^2 = a^3.$$

Hence, the condition (i) of Proposition 2.1 holds.

If the equality $a^* = a^*aa$ applies, then the proof is analogous.

(x) The conclusion follows by (iii) of Proposition 2.1 and (ix).

(xi) Applying $a^\dagger = aa^*a^*$ and $a = a^3$, we have

$$a^* = a^\dagger aa^* = a(a^*)^2aa^* = a^3(a^*)^2aa^* = a^2(a(a^*)^2)aa^* = a^2a^\dagger aa^* = a^2a^*.$$

Therefore, the condition (ix) is satisfied.

If the equalities $a = a^3$ and $a^\dagger = a^*a^*a$ hold, then the proof is analogous.

(xii) The equality $a^\dagger a = aa$ gives

$$a^* = a^*aa^\dagger = a^*a^2.$$

Hence, the condition (ix) holds.

If the equality $aa^\dagger = aa$ applies, then the proof is analogous.

(xiii) The conclusion follows by (i) and (xii).

(xiv) Suppose that $a^\dagger = a^\dagger aa$. Then

$$aa^\dagger = aa^\dagger aa = aa.$$

Thus, the condition (xii) is satisfied.

If the equality $a^\dagger = aaa^\dagger$ holds, then the proof is analogous.

(xv) The conclusion follows by (i) and (xiv). □

In the following theorem, we assume that the element a is both Moore-Penrose invertible and group invetible. Then, we study the conditions involving $a^\dagger$, $a^\sharp$ and a to ensure that a is a Moore-Penrose Hermitian element.

Theorem 2.2 *Let $\mathcal{R}$ be a ring with involution and $a \in \mathcal{R}^\dagger$. Then a is a Moore-Penrose Hermitian if and only if $a \in \mathcal{A}^\sharp$ and one of the following equivalent conditions holds:*

(i) $a^\dagger a = a^\sharp a^\dagger$.

(ii) $a = a^\dagger aa^\sharp$.

(iii) $a^\dagger a^\sharp = a^\sharp a$.

(iv) $a^\sharp = a^\dagger a^\sharp a^\sharp$.

Proof. (i) Multiplying $a^\dagger a = a^\sharp a^\dagger$ by a^2 from the left side, we get $aa = aa^\dagger$. Hence, the condition (xii) of Theorem 2.1 is satisfied.

(ii) Using $a = a^\dagger a a^\sharp$, we obtain $aa = a^\dagger(aa^\sharp a) = a^\dagger a$. Thus, the condition (xii) of Theorem 2.1 holds.

(iii) When we multiplying $a^\dagger a^\sharp = a^\sharp a$ by a^2 from the right side, we have $a^\dagger a = aa$. Hence, a satisfies the condition (xii) of Theorem 2.1.

(iv) Multiplying the equality $a^\sharp = a^\dagger a^\sharp a^\sharp$ by a from the right side, we obtain the condition (iii):

$$a^\dagger a^\sharp = a^\sharp a.\square$$

Finally, we prove the result involving Moore-Penrose Hermitian elements in a ring with involution.

Theorem 2.3 *Let $\mathcal{R}$ be a ring with involution and let $a \in \mathcal{R}$. If a is Moore-Penrose Hermitian and star-dagger, then a is normal.*

Proof. Using $a^\dagger = a$ and $a^\sharp a^\dagger = a^\dagger a^\sharp$, we get

$$a^* a = a^* a^3 = a^*(a^\dagger)^3 = (a^\dagger)^3 a^* = a^\dagger a^* = aa^*.$$

Thus, a is normal. $\square$

3 Conclusions

In this paper, we consider Moore-Penrose invertible or both Moore-Penrose invertible and group invertible elements in rings with involution to characterize Moore-Penrose Hermitian elements. In the theory of complex matrices various authors used an elegant respresentation of complex matrices to investigate Moore-Penrose Hermitian matrices. In this paper, we applied a purely algebraic technique, involving some characterizations of the Moore-Penrose inverse.

References

[1] J. Benítez, MoorePenrose inverses and commuting elements of C^*-algebras, J. Math. Anal. Appl. 345 (2) (2008), 766-770.

[2] A. Berman, Nonnegative Matrices which are Equal to Their Generalized Inverse, Linear Algebra and its Applications, 9 (1974) 261-265.

[3] E. Boasso, On the Moore-Penrose Inverse in C^*-algebras, Extracta Math. 21 (2) (2006), 93-106.

[4] D.S. Djordjević, V. Rakočević, Lectures on generalized inverses, Faculty of Sciences and Mathematics, University of Niš, 2008.

[5] M. Mišić, M. Tošić, Z. J. Popović, Generalized inverses of a linear combination of Moore-Penrose Hermitian matrices, Filomat, 30:11 (2016), 2965-2972.

[6] D. Mosić, D. S. Djordjević and J. J. Koliha, EP elements in rings, Linear Algebra Appl. 431 (5-7) (2009), 527-535.

[7] D. Mosić and D. S. Djordjević, Partial isometries and EP elements in rings with involution, Electronic J. Linear Algebra 18 (2009), 761-772.

[8] D. Mosić and D. S. Djordjević, Further results on partial isometries and EP elements in rings with involution, Math. Comput. Modelling 54 (1-2) (2011), 460-465.

[9] D. Mosić and D. S. Djordjević, Moore-Penrose-invertible normal and Hermitian elements in rings, Linear Alebra Appl., 431 (2009), 732-745.

[10] J.J. Koliha, Elements of C^*-algebras commuting with their Moore- Penrose inverse, Studia Math. 139 (2000), 81-90.

University of Pristina-Kosovska Mitrovica, Faculty of Sciences and Mathematics, 38220, Kosovska Mitrovica, Serbia.
E-mail address: marina.tosic@pr.ac.rs
University of Pristina-Kosovska Mitrovica, Faculty of Sciences and Mathematics, 38220, Kosovska Mitrovica, Serbia.
E-mail address: natasa.kontrec@pr.ac.rs
University of Pristina-Kosovska Mitrovica, Faculty of Sciences and Mathematics, 38220, Kosovska Mitrovica, Serbia.
E-mail address: eugen.ljajko@pr.ac.rs

Publisher
Union od Mathematicians of Macedonia – ARMAGANKA

Editor in chief
Prof. d-r Aleksa Malcheski

CIP - Каталогизација во публикација
Национална и универзитетска библиотека "Св. Климент Охридски", Скопје

51(082)

PROCEEDINGS of CODEMA 2020 = Зборник на трудови од CODEMA 2020. -
Skopje : Armaganka, 2021. - 375 стр. : илустр. ; 25 см

Текст на мак. и англ. јазик. - Фусноти кон текстот. - Библиографија кон трудовите

ISBN 978-608-4904-09-0

а) Математика -- Зборници

COBISS.MK-ID 53570309