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Some fixed point theorems in S-complete spaces

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Abstract

In this article we prove the existence and the uniqueness of a fixed point for a self-map f on a S-complete space (X, S) such that for all $x, y, z \in X$, $S(fx, fy, fz) \leq C \sum_{cyc}(S(x,y,y)+S(y,z,y)+S(x,z,x))$ for $0 \leq C < 1/6$, or $S(fx, fy, fz) \leq C \max\{S(a,b,c)|a,b,c \in \{x,y,z\}\}$ for $0 \leq C < 1$.

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1 Introduction

The Banach fixed point theorem [1] is a very simple and powerful theorem with a wide range of applications. It has been used by many authors for solving linear, nonlinear differential and integral equations and more recent in partial equations, fractional equations, dynamic systems, Cauchy boundary problems. Through the years this theorem has been generalized and extended by many authors in various ways and directions.
In 1963 Gahler [2] gave the concept of 2-metric space. We refer to [4] for fixed points in 2-metric space. Further, in 1992 Dhage [3] introduced the concept of D-metric spaces, as a modification of the concept of 2-metric space. In 2003 Mustafa and Sims in their paper [5] demonstrated that most of the claims concerning the fundamental topological properties of D-metric spaces are incorrect. They made an attempt to fix these problems in 2005 in [6] and these attempts resulted with the introduction of the concept of G-metric space. They proved the existence of fixed points of various contraction type mappings. Other authors also did some research in the area of fixed point concerning G-metric spaces, c.f. [7]. In 2007 Sedghi in [8] modified the concept of D-metric space and introduced the concept of D^*-metric space. He also proved a fixed point theorem in D^*-metric space. Later in 2012 Sedghi, Shobe and Aliouche [9] introduced the concept of S-metric space which differs from the previous type spaces and proved some fixed point theorems in S-metric spaces.

As a motivation for this research we present a simple example of the use of fixed point technique in solving Cauchy initial value problem. We start with countinious real valued function on $[a, b] \times [c, d]$. The Cauchy initial value problem consists of finding a continuously differentiable function y on $[a, b]$ satisfying the equations

$$\frac{dy}{dx} = f(x,y), \; y(x_0) = y_0. \tag{1}$$

We consider the Banach space $C[a,b]$ of countinuous real valued functions equiped with the norm $\|y\| = \sup\{y(x)|x \in [a,b]\}$. Integrating the equation (1) one obtais the following

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt. \tag{2}$$

The problem (2) is equivalent with the problem (1). We define the operator $T : C[a,b] \to C[c,d]$ with

$$Ty = y_0 + \int_{x_0}^{x} f(t, y(t))dt. \tag{3}$$

Hence solving (3) reduces to finding fixed point of the operator T.

The main problem of finding the fixed points of similarly defined operators is to find conditions concerning the operator T such that it becomes some kind of contraction.

2 Preliminaries

The notation, definitions and elementary results given in this section are from [9]. We give the basic definitions concerning S-metric space, i.e. S-convergent sequence, S-Cauchy sequence and S-complete space, then some known examples and basic properties of S-metric spaces.

Definition 2.1. *Let X be a nonempty set. A function $S : X^3 \to \mathbb{R}_0^+$ is called an S-metric on X, if for each $x, y, z, a \in X$ the following conditions are satisfied*

$(S1)$ $S(x,y,z) = 0$ if and only if $x = y = z$; and

$(S2)$ $S(x,y,z) \le S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) is called an S-metric space.

Lemma 2.2. *([9]) Let (X,S) be an S-metric space. Then $S(x,x,y) = S(y,y,x)$, for all $x, y \in X$.*

Definition 2.3. *A sequence $(x_n)_{n=1}^{\infty}$ in a S-metric space (X,S) is called S-convergent, if there is $x \in X$ such that $S(x_n, x_n, x) \to 0$, as $n \to \infty$.*

Definition 2.4. *A sequence $(x_n)_{n=1}^{\infty}$ in a S-metric space (X,S) is called S-Cauchy, if $S(x_n, x_n, x_m) \to 0$, as $n, m \to \infty$.*

Definition 2.5. *An S-metric space (X,S) is called S-complete, if every S-Cauchy sequence in X converges in X.*

Concerning the topology one has the following.

Definition 2.6. *([9]) Let (X, S) be an S-metric space. For $x \in X$ and $r > 0$, the open ball $B_S(x, r)$ is the set*

$$B_S(x, r) = \{y \in X \, : \, S(y, y, x) < r\}.$$

Let τ be the set of all $A \subset X$ such that for all $x \in A$ there exist $r > 0$ and $B_S(x, r) \subset A$. Then τ is a topology on X induced by the S-metric S.

Lemma 2.7. *([9]) Let (X, S) be an S-metric space. For $x \in X, r > 0$ the ball $B_S(x, r)$ is an open subset of X.*

Trivially but worth mentioning is the fact that the sequence convergence in (X, S) is equivalent with τ convergence, where τ is induced by the S-metric S.

Example 2.8. *Let $X = \mathbb{R}^n$ and $\| \cdot \|$ be a norm on X. Then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X.*

Example 2.9. *Let X be a nonempty set and d be an ordinary metric on X. Then $S(x, y, z) = d(x, z) + d(y, z)$ is an S-metric on X.*

Example 2.10. *Let $X = [0, 1]$. We define $S : X^3 \to \mathbb{R}_0^+$ by:*

$$S(x, y, z) = \begin{cases} 0 & , x = y = z \\ \max\{x, y, z\} & , \textit{otherwise} \end{cases} .$$

Then S is an S-metric on X.

Example 2.11. *Let X be a nonempty set and d_1, d_2 are two metrics on X. Then $S(x, y, z) = d_1(x, z) + d_2(y, z)$ is an S-metric on X.*

Example 2.12. *If X is a vector space over $\mathbb{R}$ and $\| \cdot \|$ is a norm on X, then $S(x, y, z) = \|\alpha y + \beta z - \gamma x\| + \|y - z\|$, where $\alpha + \beta = \gamma$ for every $\alpha, \beta \geq 1$, is an S-metric on X.*

3 Main results

The main idea of this article is to define certain self-mapings on (X, S) and to prove the existence and the uniqueness of their fixed points.

Theorem 3.1. *Let f be a self-map on an S-complete space (X, S) such that $S(fx, fy, fz) \leq C \sum_{cyc}(S(x, y, y) + S(y, z, y) + S(x, z, x))$ for all $x, y, z \in X$, where $0 \leq C < 1/6$. Then f has a unique fixed point. The sum is over all cyclic permutations of the triple (x, y, z).*

Proof. Plugging $y = x$ in to the condition for the mapping f and one obtains

$$S(fx, fx, fz) \leq 2C(S(x, z, x) + S(z, x, z)) + C(S(x, z, z) + S(z, x, x)). \quad (4)$$

Directly from $(S2)$ in the definition (2.1) it follows that $S(x, y, x) \le S(y, y, x)$ and $S(x, y, y) \le S(x, x, y)$ for all $x, y \in X$. Applying these inequalities in (4) we obtain,

$$S(fx, fx, fz) \le 2C(S(z, z, x) + S(x, x, z)) + C(S(x, x, z) + S(z, z, x)).$$

Lemma 2.2 implies

$$S(fx, fx, fz) \le 6CS(x, x, z) = MS(x, x, z). \tag{5}$$

where $0 \le 6C = M < 1$.

Let $x_0 \in X$ be an arbitrary point. We take the orbit of x_0, i.e. the sequence $(x_n)_{n=1}^{\infty}$ defined with the conditions: $x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), \ldots, x_n = f^n(x_0), \cdots$. For simplicity, we use the notation fx_n for $f(x_n)$. Then we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\le MS(x_{n-1}, x_{n-1}, x_n) \le M^2 S(x_{n-2}, x_{n-2}, x_{n-1}) \\ &\le M^n S(x_0, x_0, x_1). \end{aligned}$$

Thus, since $0 \le M < 1$,

$$\lim_{n \to \infty} S(x_n, x_n, x_{n+1}) = 0. \tag{6}$$

Next, we will show that the sequence $(x_n)_{n=1}^{\infty}$ is S-Cauchy. From the definition of $(x_n)_{n=1}^{\infty}$ and $(S2)$ we obtain

$$\begin{aligned} S(x_n, x_n, x_m) &\le S(x_n, x_n, fx_n) + S(x_n, x_n, fx_n) + S(x_m, x_m, fx_n) \\ &= 2S(x_n, x_n, fx_n) + S(x_m, x_m, fx_n) \\ &\le 2S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m) + S(x_m, x_m, fx_m) + S(fx_n, fx_n, fx_m) \\ &= 2[S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m)] + S(fx_n, fx_n, fx_m). \end{aligned}$$

Choosing $x = x_n$ and $z = x_m$ in inequality (5), one obtains $S(fx_n, fx_n, fx_m) \le MS(x_n, x_n, x_m)$, hence

$$S(x_n, x_n, x_m) \le 2[S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m)] + MS(x_n, x_n, x_m).$$

$$(1 - M)S(x_n, x_n, x_m) \le 2[S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m)],$$

i.e.

$$S(x_n, x_n, x_m) \le 2/(1 - M)[S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m)].$$

From (6) it follows that $S(x_n, x_n, x_m) \to 0$, as $n, m \to \infty$, i.e. the sequence $(x_n)_{n=1}^{\infty}$ is S-Cauchy. Since (X, S) is S-complete, there exists $p \in X$ such that

$$\lim_{n \to \infty} S(x_n, x_n, p) = 0. \tag{7}$$

Next we will prove that $p \in X$ is a fixed point for f. Using $(S2)$ and Lemma 2.2 we obtain

$$\begin{aligned} S(fp, fp, p) &\le S(fp, fp, fx_n) + S(fp, fp, fx_n) + S(p, p, fx_n) \\ &= 2S(fp, fp, fx_n) + S(p, p, fx_n) \\ &= 2S(fx_n, fx_n, fp) + S(fx_n, fx_n, p). \end{aligned}$$

We choose $x = x_n$ and $z = p$ in (5) and obtain $S(fx_n, fx_n, fp) \le MS(x_n, x_n, p)$. Hence, $S(fp, fp, p) \le 2MS(x_n, x_n, p) + S(fx_n, fx_n, p)$. On the other hand, $S(fx_n, fx_n, p) \le 2S(fx_n, fx_n, x_n) + S(p, p, x_n) = 2S(x_n, x_n, fx_n) + S(x_n, x_n, p)$. Thus,

$$S(fp, fp, p) \le (2M + 1)S(x_n, x_n, p) + 2S(x_n, x_n, x_{n+1}). \qquad (8)$$

As $n \to \infty$, from (6) and (7) it follows that $S(fp, fp, p) = 0$, i.e. $fp = p$. Next we will show the uniqueness of the fixed point p. Let $q \in X$ be another fixed point for f. Then we choose $x = p$ and $z = q$ in (5) and obtain $S(p, p, q) = S(fp, fp, fq) \le MS(p, p, q)$, i.e. $(1 - M)S(p, p, q) \le 0$. From $0 \le M < 1$ it follows that $S(p, p, q) = 0$. Thus, $p = q$. □

Theorem 3.2. *Let f be a self-map on an S-complete space (X, S) such that $S(fx, fy, fz) \le C \max\{S(a, b, c) | a, b, c \in \{x, y, z\}\}$ for all $x, y, z \in X$, where $0 \le C < 1$. Then f has a unique fixed point.*

Proof. If we set $y = x$ in the inequality defining f we obtain

$$\begin{aligned} S(fx, fx, fz) &\le C \max\{S(a, b, c) | a, b, c \in \{x, z\}\} \\ &= C \max\{S(x, x, x), S(x, z, z), S(z, x, z), S(z, z, x), \\ &\quad S(x, x, z), S(x, z, x), S(z, x, x), S(z, z, z)\} \end{aligned}$$

From Lemma 2.2 and $(S2)$ we obtain $S(x, z, z) \le S(x, x, z), S(z, x, z) \le S(x, x, z)$, $S(x, z, x) \le S(z, z, x) = S(x, x, z)$ and $S(z, x, x) \le S(z, z, x) = S(x, x, z)$. Hence, $S(fx, fx, fz) \le CS(x, x, z)$. The rest of the proof is analogous to the previous theorem.

□

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