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COMMON FIXED POINTS FOR TWO T_f KANNAN TYPE CONTRACTIONS IN A COMPLETE METRIC SPACE

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Abstract. The focus in this paper are statements about common fixed points for two T_f Kannan type contractions in a complete metric space (X,d). In doing so we defined T as continuous, injection and subsequentially convergent mapping, and f as a function of the class Θ continuous monotony non-decreasing functions $f:[0,+\infty)\to[0,+\infty)$ such that $f^{-1}(0)=\{0\}$, and furthermore f is sub-additive i.e. $f(x+y)\le f(x)+f(y)$, for all $x,y\in[0,+\infty)$.

1. INTRODUCTION

The Banach fixed-point theorem, as well as its generalizations presented by R. Kannan ([4]), S. K. Chatterjea ([7]) and P. V. Koparde, B. B. Waghmode ([3]), are well known. S. Moradi and D. Alimohammadi [9] generalized R. Kannan results using the sequentially convergent mappings. Using the sequentially convergent mappings, some generalizations of R. Kannan, S. K. Chatterjea and P. V. Koparde, B. B. Waghmode are proved [1], and also proved results about sharing fixed point for two R. Kannan, S. K. Chatterjea и P. V. Koparde, B. B. Waghmode types of mapping [5], 2006.

S. Moradi and A. Beiranvand introduced the concept of T_f contractive mapping, [8], 2010, applying the Θ class of continuous monotony non-decreasing functions $f:[0,+\infty)\to[0,+\infty)$ such that $f^{-1}(0)=\{0\}$. For $f\in\Theta$, $f^{-1}(0)=\{0\}$ implies that $f(t)>0$, for all $t>0$. S. Moradi and A.Beiranvand proved that if S is T_f contractive mapping, then S has a unique fixed point. M. Kir and H. Kiziltunc, 2014 generalized the S. Moradi and A. Beiranvand result about R. Kannan and S. K. Chatterjea types of mapping.

In our further observations we will present several results about sharing fixed points of two T_f Kannan type contractions in a complete metric space, such that the function f, f is a function of Θ class, and additionally we will suppose that it is subadditive, i.e. $f(a+b)\le f(a)+f(b)$, for all $a,b\in[0,+\infty)$.

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2. Main Results

Definition 1 ([8]). Let (X,d) be a metric space. A mapping $T:X\to X$ is said sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ also is convergence. A mapping T is said sub-sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ has a convergent subsequence.

Definition 2 ([8]). Let (X,d) be a metric space, $S,T:X\to X$ and $f\in\Theta$. A mapping S is said T_f $-$contraction if there exist $\alpha\in(0,1)$ such that

$$f(d(TSx,TSy))\le\alpha f(d(Tx,Ty)),$$

for all $x,y\in X$.

Theorem 1. Let (X,d) be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ is such that $f(a+b)\le f(a)+f(b)$, for all $a,b\in[0,+\infty)$ and the mapping $T:X\to X$ be continuous, injection and subsequentially convergent. If there exist $\alpha>0,\beta\ge 0$ such that $2\alpha+\beta\in(0,1)$ and

$$f(d(TS_1x,TS_2y))\le\alpha(f(d(Tx,TS_1x))+f(d(Ty,TS_2y)))+\beta f(d(Tx,Ty)) \quad (1)$$

for all $x,y\in X$, then S_1 and S_2 have a unique sharing fixed point.

Proof. Let x_0 be any point of X and let the sequence $\{x_n\}$ be defined as

$$x_{2n+1}=S_1x_{2n},\ x_{2n+2}=S_2x_{2n+1},\ n=0,1,2,3,\dots .$$

If there exists $n\ge 0$, such that $x_n=x_{n+1}=x_{n+2}$, then it is easy to be proven that $u=x_n$ is a sharing fixed point for S_1 and S_2. Therefore, let us assume that there no exist three consecutive equal terms of the sequence $\{x_n\}$. Then by applying the inequality (1) , it is easy to prove the following inequalities:

$$f(d(Tx_{2n+1},Tx_{2n}))\le\alpha(f(d(Tx_{2n+1},Tx_{2n}))+f(d(Tx_{2n-1},Tx_{2n})))+ \\ +\beta f(d(Tx_{2n},Tx_{2n-1}))$$

and

$$f(d(Tx_{2n-1},Tx_{2n}))\le\alpha(f(d(Tx_{2n-2},Tx_{2n-1}))+f(d(Tx_{2n-1},Tx_{2n})))+ \\ +\beta f(d(Tx_{2n-2},Tx_{2n-1})).$$

The above stated implies that for each $n=0,1,2,\dots$ and $\lambda=\frac{\alpha+\beta}{1-\alpha}\in(0,1)$ holds:

$$f(d(Tx_{n+1},Tx_n))\le\lambda f(d(Tx_n,Tx_{n-1})). \quad (2)$$

Moreover, the inequality (2) implies

$$f(d(Tx_{n+1},Tx_n))\le\lambda^n f(d(Tx_1,Tx_0)), \quad (3)$$

for each $n=0,1,2,\dots$.So, the properties of metrics, the properties of the function f and the inequality (3) imply

$$f(d(Tx_n,Tx_m)) \le \frac{\lambda^m}{1-\lambda} f(d(Tx_1,Tx_0)),$$

for all $m,n \in \mathbb{N}$, $n > m$. According to this, the sequence $\{Tx_n\}$ is Caushy sequence, and since (X,d) is a complete metric space, it is convergent sequence. The mapping $T: X \to X$ is subsequentially convergent, therefore the sequence $\{x_n\}$ consists a convergent sub-sequence $\{x_{n(k)}\}$, i.e. it exists $u \in X$ such that $\lim_{k\to\infty} x_{n(k)} = u$. The continuity of T implies that $\lim_{k\to\infty} Tx_{n(k)} = Tu$. But, $\{Tx_{n(k)}\}$ is a subsequence of the convergent sequence $\{Tx_n\}$. So,

$$\lim_{n\to\infty} Tx_n = \lim_{k\to\infty} Tx_{n(k)} = Tu.$$

Next, we will prove that $u \in X$ is a fixed point for the mapping S_1.

$$\begin{aligned} f(d(Tu,TS_1u)) &\le f(d(Tu,Tx_{2n+2})) + f(d(Tx_{2n+2},TS_1u)) \\ &= f(d(Tu,Tx_{2n+2})) + f(d(TS_2x_{2n+1},TS_1u)) \\ &\le f(d(Tu,Tx_{2n+2})) + \alpha(f(Tu,TS_1u) + f(d(Tx_{2n+1},TS_2x_{2n+1}))) + \\ &\quad + \beta f(d(Tu,Tx_{2n+1})) \\ &= f(d(Tu,Tx_{2n+2})) + \alpha(f(Tu,TS_1u) + f(d(Tx_{2n+1},Tx_{2n+2}))) + \\ &\quad + \beta f(d(Tu,Tx_{2n+1})) \end{aligned}$$

The mappings f and T are continuous, and applying the properties of metric for $n \to \infty$ in the last inequality, we get that

$$f(d(Tu,TS_1u)) \le \alpha f(Tu,TS_1u) + (1+\alpha+\beta) f(0)$$

But, $1-\alpha > 0$ and $f^{-1}(0) = \{0\}$, therefo.re the last inequality implies $d(Tu,TS_1u) = 0$, that is $TS_1u = Tu$. Finally, since T is injection it holds that $S_1u = u$, that u is a fixed point for the mapping S_1. Analogously, u is fixed point for S_2, i.e. u is a sharing fixed point for the both S_1 and S_2 mapping.

Further, we will prove that S_1 and S_2 have the unique sharing fixed point. Let $v \in X$ be fixed point for S_2, i.e. $S_2v = v$. Then

$$\begin{aligned} f(d(Tu,Tv)) &= f(d(TS_1u,TS_2v)|) \\ &\le \alpha(f(d(Tu,TS_1u)) + f(d(Tv,TS_2v))) + \beta f(d(Tu,Tv) \\ &= \alpha(f(d(Tu,Tu)) + f(d(Tv,Tv))) + \beta f(d(Tu,Tv) \\ &= 2\alpha f(0) + \beta f(d(Tu,Tv). \end{aligned}$$

And since $1-\beta > 0$ and $f^{-1}(0) = \{0\}$, applying the last inequality we get that $d(Tu,Tv) = 0$, i.e. it holds that $Tu = Tv$.

But since T is injection, we get that $u = v$, that is S_1 and S_2 have a unique sharing fixed point. ■

Corollary 1. Let (X,d) be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ is such that $f(a+b)\le f(a)+f(b)$, for all $a,b\in[0,+\infty)$ and the mapping $T:X\to X$ be continuous, injection and subsequentially convergent. If it exists $\lambda\in(0,1)$ such that

$$f(d(TS_1x,TS_2y))\le\lambda\sqrt[3]{f(d(Tx,TS_1x))\cdot f(d(Ty,TS_2y))\cdot f(d(Tx,Ty)}$$

for all $x,y\in X$, then S_1 and S_2 have an unique sharing fixed point.

Proof. The arithmetic-geometric mean inequality implies that

$$f(d(TS_1x,TS_2y))\le\tfrac{\lambda}{3}(f(d(Tx,TS_1x))+f(d(Ty,TS_2y))+f(d(Tx,Ty))),$$

for all $x,y\in X$. Applying the Theorem 1 for $\alpha=\beta=\frac{\lambda}{3}$ we get the above corollary. ∎

Corollary 2. Let (X,d) be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ be such that $f(a+b)\le f(a)+f(b)$,for all $a,b\in[0,+\infty)$ and the mapping $T:X\to X$ be continuous, injection and subsequentially convergent. If there exist $\alpha>0,\beta\ge0$ such that $2\alpha+\beta\in(0,1)$ and

$$f(d(TS_1x,TS_2y))\le\alpha\frac{f^2(d(Tx,TS_1x))+f^2(d(Ty,TS_2y))}{f(d(Tx,TS_1x))+f(d(Ty,TS_2y))}+\beta f(d(Tx,Ty),$$

for all $x,y\in X$, then S_1 and S_2 have a unique sharing fixed point.

Proof. The inequality (1) is a direct implication of the given inequality. ∎

Corollary 3. Let (X,d) be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ be such that $f(a+b)\le f(a)+f(b)$,for all $a,b\in[0,+\infty)$ and the mapping $T:X\to X$ be continuous, injection and subsequentially convergent. If it exists $\alpha\in(0,\frac{1}{2})$ such that

$$f(d(TS_1x,TS_2y))\le\alpha(f(d(Tx,TS_1x))+f(d(Ty,TS_2y))),$$

for all $x,y\in X$, then S_1 and S_2 have a unique sharing fixed point.

Proof. The proof is directly implied by Theorem 1, for $\beta=0$. ∎

Corollary 4. Let (X,d) be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ be such that $f(a+b)\le f(a)+f(b)$,for all $a,b\in[0,+\infty)$. If there exist $\alpha>0,\beta\ge0$ so that $2\alpha+\beta\in(0,1)$ and

$$f(d(S_1x,S_2y))\le\alpha(f(d(x,S_1x))+f(d(y,S_2y)))+\beta f(d(x,y))$$

for all $x,y\in X$, then S_1 and S_2 have a unique sharing fixed point.

Proof. The mapping $T:X\to X$ determined as $Tx=x$ е непрекинато, инјекција и subsequentially convergent.

Therefore, the proof is directly implied by Theorem 1, for $Tx=x$. ∎

Corollary 5. Let (X,d) be a complete metric space $S_1,S_2 : X \to X$, $f \in \Theta$ be such that $f(a+b) \le f(a)+f(b)$,for all $a,b \in [0,+\infty)$. If it exists $\alpha \in (0,\frac{1}{2})$ so that

$f(d(S_1x,S_2y)) \le \alpha(f(d(x,S_1x))+f(d(y,S_2y)))$, for all $x,y \in X$ it holds , then S_1 and S_2 have a unique sharing fixed point.

Proof. Direct implication from the Corollary 3 for $Tx = x$ or the Corollary 4 for $\beta = 0$. ■

Corollary 6. Let (X,d) be a complete metric space $S_1,S_2 : X \to X$, $f \in \Theta$ be such that $f(a+b) \le f(a)+f(b)$,for all $a,b \in [0,+\infty)$ and the mapping $T : X \to X$ be continuous, injection and subsequentially convergent. If there exist $p,q \in \mathbb{N}$ and $\alpha > 0, \beta \ge 0$ such that $2\alpha + \beta \in (0,1)$ and

$$f(d(TS_1^p x, TS_2^q y)) \le \alpha(f(d(Tx,TS_1^p x))+f(d(Ty,TS_2^q y)))+\beta f(d(Tx,Ty))$$

for all $x,y \in X$, then S_1 and S_2 have a unique sharing fixed point.

Proof. The Theorem1 implies that the mappings S_1^p and S_2^q have a unique common fixed point $u \in X$. So, $S_1^p u = u$ and therefore

$$S_1u = S_1(S_1^p u) = S_1^p(S_1u),$$

that is S_1u is a fixed point for S_1^p . Analogously, $S_2^q u = u$ implies that

$$S_2u = S_2(S_2^q u) = S_2^q(S_2u),$$

that is S_2u a fixed point for S_2^q . But, the proof of the Theorem1 implies that S_1^p and S_2^q *has* unique fixed points. Therefore $S_1u = u$ and $S_2u = u$. According to this, $u \in X$ is a common fixed point for S_1 and S_2 .

For $v \in X$ is an arbitrary fixed point for S_1 and S_2 , we get that it is also a common fixed point for S_1^p and S_2^q . But the mappings S_1^p and S_2^q have a unique common fixed point, and therefore $v = u$. ■

Remark. The function $f:[0,+\infty) \to [0,+\infty)$ defined as $f(t) = t$, for each $t \in [0,1)$, it is a function of Θ class and it is a subadditive. Moreover, each sequentially convergent mapping $T : X \to X$ is sub-sequentially convergent mapping. Therefore the Theorem1 and the Corollaries 1-6 [5], are directly implied by the above proved the Theorem 1 and the Corollaries 1-6, respectively.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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