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**OSTROWSKI-GRÜSS TYPE INEQUALITY OF CHEBYSHEV
FUNCTIONAL WITH APPLICATION TO THE WEIGHT
THREEPOINT INTEGRAL FORMULA**

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S. KOVAČ

ABSTRACT. Recently there have been proven many results about error bounds for Chebyshev functional. The aim of our paper is to extend those results and give some new error estimation of the Chebyshev functional and applications to the threepoint weighted integral formulas.

1. INTRODUCTION

For two Lebesgue integrable functions $f, g : [a, b] \rightarrow \mathbb{R}$ let us consider the Chebyshev functional:

$$T(f, g) := \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{b-a} \int_a^b f(t)dt \cdot \frac{1}{b-a} \int_a^b g(t)dt. \quad (1.1)$$

$L_p[a, b]$, $1 \leq p < \infty$ stands for the space of the functions $f : [a, b] \rightarrow \mathbb{R}$ which are p -integrable, i.e. they are equipped by p -norm

$$\|f\|_p = \left[\int_a^b |f(t)|^p dt \right]^{\frac{1}{p}}$$

which is finite. $L_\infty[a, b]$ stands for space of the functions $f : [a, b] \rightarrow \mathbb{R}$ which are essentially bounded and ∞ -norm defined by

$$\|f\|_\infty = \text{esssup}_{t \in [a, b]} |f(t)|$$

is finite.

P. Cerone and S.S. Dragomir have delivered in [1] the following bounds for Chebyshev functional $T(\varphi, \varphi)$:

Lemma 1. *If $\varphi : [a, b] \rightarrow \mathbb{R}$ is an absolutely continuous function with*

$$(\cdot - a)(b - \cdot)(\varphi')^2 \in L[a, b],$$

then we have the inequality

$$T(\varphi, \varphi) \leq \frac{1}{2(b-a)} \int_a^b (x-a)(b-x) [\varphi'(x)]^2 dx. \quad (1.2)$$

The constant $\frac{1}{2}$ is the best possible.

The following results of Grüss type have been obtained in [1]:

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Theorem 1. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two absolutely continuous functions on $[a, b]$ with

$$(\cdot - a)(b - \cdot)(f')^2, (\cdot - a)(b - \cdot)(g')^2 \in L[a, b].$$

Then we have the inequality

$$\begin{aligned} |T(f, g)| &\leq \frac{1}{\sqrt{2}} [T(f, f)]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}} \left(\int_a^b (x-a)(b-x) [g'(x)]^2 dx \right)^{\frac{1}{2}} \quad (1.3) \\ &\leq \frac{1}{2(b-a)} \left(\int_a^b (x-a)(b-x) [f'(x)]^2 dx \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_a^b (x-a)(b-x) [g'(x)]^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

The constant $\frac{1}{\sqrt{2}}$ and $\frac{1}{2}$ are best possible in (1.3).

Theorem 2. Assume that $g : [a, b] \rightarrow \mathbb{R}$ is monotonic nondecreasing on $[a, b]$ and $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous with $f' \in L_\infty[a, b]$. Then we have the inequality

$$|T(f, g)| \leq \frac{1}{2(b-a)} \|f'\|_\infty \int_a^b (x-a)(b-x) dg(x). \quad (1.4)$$

The constant $\frac{1}{2}$ is the best possible.

In this paper we shall give the notion and new bounds for general open weighted threepoint formula by using upper results for Chebyshev functional.

2. GENERAL OPEN WEIGHT THREEPOINT INTEGRAL FORMULA

Let us recall the general integral formula obtained in [2]. Consider subdivision $\sigma = \{a = x_0 < x_1 < \dots < x_m = b\}$ of the segment $[a, b]$, for some $m \in \mathbb{N}$. Let $w : [a, b] \rightarrow \mathbb{R}$ be an arbitrary integrable function. On each interval $[x_{k-1}, x_k]$, $k = 1, \dots, m$ we consider different w -harmonic sequences of functions $\{w_{kj}\}_{j=1, \dots, n}$, i.e. we have

$$\begin{aligned} w'_{k1}(t) &= w(t) && \text{for } t \in [x_{k-1}, x_k] \\ (w_{kj})'(t) &= w_{k, j-1}(t) && \text{for } t \in [x_{k-1}, x_k], \text{ for all } j = 2, 3, \dots, n. \end{aligned} \quad (2.1)$$

Further, let us define

$$W_{n,w}(t, \sigma) = \begin{cases} w_{1n}(t) & \text{for } t \in [a, x_1], \\ w_{2n}(t) & \text{for } t \in (x_1, x_2], \\ \vdots & \\ w_{mn}(t) & \text{for } t \in (x_{m-1}, b]. \end{cases} \quad (2.2)$$

In order to obtain the weight threepoint integral formula, we shall use the following results about the general integral formula ([2]):

Theorem 3. *If $g : [a, b] \rightarrow \mathbb{R}$ is such that $g^{(n)}$ is a piecewise continuous on $[a, b]$, then the following identity holds*

$$\int_a^b w(t)g(t)dt = \sum_{j=1}^n (-1)^{j-1} [w_{mj}(b)g^{(j-1)}(b) + \sum_{k=1}^{m-1} [w_{kj}(x_k) - w_{k+1,j}(x_k)]g^{(j-1)}(x_k) - w_{1j}(a)g^{(j-1)}(a)] + (-1)^n \int_a^b W_{n,w}(t, \sigma)g^{(n)}(t)dt. \tag{2.3}$$

Theorem 4. *Let us suppose $w : [a, b] \rightarrow \mathbb{R}$ is an integrable function and $\{w_{kj}\}_{j=1, \dots, n}$ w -harmonic sequences of functions on $[x_{k-1}, x_k]$, for $k = 1, \dots, m$. If $g : [a, b] \rightarrow \mathbb{R}$ is a function such that $g^{(n-1)}$ is absolutely continuous and $g^{(n)} \in L_p$ for some $1 \leq p \leq \infty$, then the following inequality holds*

$$\left| \int_a^b w(t)g(t)dt - \sum_{j=1}^n (-1)^{j-1} [w_{mj}(b)g^{(j-1)}(b) + \sum_{k=1}^{m-1} [w_{kj}(x_k) - w_{k+1,j}(x_k)]g^{(j-1)}(x_k) - w_{1j}(a)g^{(j-1)}(a)] \right| \leq C(n, p, w) \cdot \|g^{(n)}\|_q, \tag{2.4}$$

where

$$C(n, p, w) = \begin{cases} \left[\int_a^b |W_{n,w}(t, \sigma)|^q dt \right]^{\frac{1}{q}}, & 1 < p \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = 1 \\ \sup_{t \in [a, b]} |W_{n,w}(t, \sigma)|, & p = 1. \end{cases} \tag{2.5}$$

The inequality is the best possible for $p = 1$ and sharp for $1 < p \leq \infty$. Equality is attained for every function g such that

$$g(t) = M \cdot g_*(t) + p_{n-1}(t),$$

where $M \in \mathbb{R}$, p_{n-1} is an arbitrary polynomial of degree at most $n - 1$ and $g_*(t)$ is function on $[a, b]$ defined by

$$g_*(t) = \int_a^t \frac{(t - \xi)^{n-1}}{(n - 1)!} \cdot \operatorname{sgn} W_{n,w}(\xi, \sigma) \cdot |W_{n,w}(\xi, \sigma)|^{\frac{1}{p-1}} d\xi, \tag{2.6}$$

for $1 < p < \infty$, and

$$g_*(t) = \int_a^t \frac{(t - \xi)^{n-1}}{(n - 1)!} \cdot \operatorname{sgn} W_{n,w}(\xi, \sigma) d\xi, \tag{2.7}$$

for $p = \infty$.

We shall deliver now the general threepoint integral fomula with nodes $x, \frac{a+b}{2}$ and $a+b-x$, for some $x \in [a, \frac{a+b}{2})$. Let $n \in \mathbb{N}$ and $\{L_j\}_{j=0,1, \dots, n}$ be some sequence of harmonic polynomials such that $\deg L_j \leq j - 1$ and $L_0 \equiv 0$. Let us consider subdivision of the segment $[a, b]$:

$$\sigma := \{x_0 < x_1 < x_2 < x_3 < x_4\}$$

where $x_0 = a$, $x_1 = x$, $x_2 = \frac{a+b}{2}$, $x_3 = a + b - x$ and $x_4 = b$. For $k = 1, \dots, n$ we define functions $w_{jk}^{(3)} : [x_{j-1}, x_j] \rightarrow \mathbf{R}$, for $j = 1, 2, 3, 4$, in the following way:

$$\begin{aligned} w_{1k}^{(3)}(t) &:= \frac{1}{(k-1)!} \int_a^t (t-s)^{k-1} w(s) ds, \\ w_{2k}^{(3)}(t) &:= \frac{1}{(k-1)!} \int_x^t (t-s)^{k-1} w(s) ds + L_k(t), \\ w_{3k}^{(3)}(t) &:= -\frac{1}{(k-1)!} \int_t^{a+b-x} (t-s)^{k-1} w(s) ds + (-1)^k L_k(a+b-t), \\ w_{4k}^{(3)}(t) &:= -\frac{1}{(k-1)!} \int_t^b (t-s)^{k-1} w(s) ds, \end{aligned}$$

and $w_{j0}^{(3)}(t) := w(t)$. Further, let us define

$$W_{n,w}^{(3)}(t, x) = \begin{cases} w_{1n}^{(3)}(t), & \text{for } t \in [a, x], \\ w_{2n}^{(3)}(t), & \text{for } t \in (x, \frac{a+b}{2}], \\ w_{3n}^{(3)}(t), & \text{for } t \in (\frac{a+b}{2}, a+b-x] \\ w_{4n}^{(3)}(t), & \text{for } t \in (a+b-x, b]. \end{cases} \tag{2.8}$$

Remark 1. Sequences $\{w_{jk}^{(3)}\}_{k=0,1,\dots,n}$ are w -harmonic sequences of functions on $[x_{j-1}, x_j]$, for every $j = 1, 2, 3, 4$.

Remark 2. If, in addition, we have $w(t) = w(a+b-t)$, for each $t \in [a, b]$, then the following symmetry conditions hold for $k = 1, \dots, n$:

$$w_{1k}^{(3)}(t) = (-1)^k w_{4k}^{(3)}(a+b-t), \quad \text{for } t \in [a, x]$$

and

$$w_{2k}^{(3)}(t) = (-1)^k w_{3k}^{(3)}(a+b-t), \quad \text{for } t \in (x, \frac{a+b}{2}].$$

Theorem 5. Let $f : [a, b] \rightarrow \mathbf{R}$ be a function with piecewise continuous n -th derivative, for some $n \in \mathbf{N}$, $w : [a, b] \rightarrow \mathbf{R}$ integrable function such that $w(t) = w(a+b-t)$, for each $t \in [a, b]$ and $x \in [a, \frac{a+b}{2})$. Further, let $\{L_k\}_{k=0,1,\dots,n}$ be some sequence of harmonic polynomials such that $\deg L_j \leq j-1$ and $L_0(t) = 0$, and $W_{n,w}^{(3)}(t, x)$ be defined by (2.8). Then the following identity holds:

$$\begin{aligned} \int_a^b w(t)f(t)dt &= A_{1,w}^{(3)}(x) (f(x) + f(a+b-x)) + B_{1,w}^{(3)}(x) f\left(\frac{a+b}{2}\right) \\ &+ T_{n,w}^{(3)}(x) + (-1)^n \int_a^b W_{n,w}^{(3)}(t, x) f^{(n)}(t) dt, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} T_{n,w}^{(3)}(x) &= \sum_{k=2}^n A_k(x) \left(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a+b-x) \right) \\ &+ \sum_{\substack{k=3 \\ \text{odd } k}} B_k(x) f^{(k-1)}\left(\frac{a+b}{2}\right), \end{aligned} \tag{2.10}$$

$$A_{k,w}^{(3)}(x) = (-1)^{k-1} \left[\frac{1}{(k-1)!} \int_a^x (x-s)^{k-1} w(s) ds - L_k(x) \right], \quad k \geq 1,$$

$$B_{k,w}^{(3)}(x) = 2 \left[\frac{1}{(k-1)!} \int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - s\right)^{k-1} w(s) ds + L_k\left(\frac{a+b}{2}\right) \right], \quad \text{for odd } k \geq 1$$

and

$$B_{k,w}^{(3)}(x) = w_{2k}\left(\frac{a+b}{2}\right) - w_{3k}\left(\frac{a+b}{2}\right) = 0, \quad \text{for even } k \geq 1.$$

Proof. We consider subdivision $x_0 = a, x_1 = x, x_2 = \frac{a+b}{2}, x_3 = a + b - x$ and $x_4 = b$ of the interval $[a, b]$ and apply formula (2.3) with $m = 4$. □

Theorem 6. Let $f : [a, b] \rightarrow \mathbb{R}$ be a function with a piecewise continuous n -th derivative and $f^{(n)} \in L_p[a, b]$ for some $n \in \mathbb{N}$ and some $1 \leq p \leq \infty$. Then we have the following inequality

$$\begin{aligned} & \left| \int_a^b w(t)f(t)dt - A_{1,w}^{(3)}(x)(f(x) + f(a+b-x)) - B_{1,w}^{(3)}(x)f\left(\frac{a+b}{2}\right) - T_{n,w}^{(3)}(x) \right| \\ & \leq C_3(n, p, x, w) \cdot \|f^{(n)}\|_p, \end{aligned} \tag{2.11}$$

where

$$C_3(n, p, x, w) = \begin{cases} 2^{1/q} \left[\int_a^{\frac{a+b}{2}} |W_{n,w}^{(3)}(t, x)|^q dt \right]^{1/q}, & \frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p \leq \infty \\ \sup\{|W_{n,w}^{(3)}(t, x)|, t \in [a, \frac{a+b}{2}]\}, & p = 1, \end{cases} \tag{2.12}$$

The inequality is best possible for $p = 1$ and sharp for $1 < p \leq \infty$. Equality is attained for the function $f_* : [a, b] \rightarrow \mathbb{R}$ defined by

$$f_*(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} |W_{n,w}^{(3)}(s, x)|^{\frac{1}{p-1}} \operatorname{sgn} W_{n,w}^{(3)}(s, x) ds \tag{2.13}$$

for $1 < p < \infty$, while for $p = \infty$

$$f_*(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} \operatorname{sgn} W_{n,w}^{(3)}(s, x) ds \tag{2.14}$$

Proof. The proof follows from the Theorem 4 for $m = 4$ and subdivision $x_0 = a, x_1 = x, x_2 = \frac{a+b}{2}, x_3 = a + b - x$ and $x_4 = b$. □

Remark 3. Using Theorem 5 and Theorem 6 for the uniform weight $w(t) = \frac{1}{b-a}$, we get the results obtained in [3], so these results are the generalization of the non-weight threepoint formula.

Let us consider the special harmonic polynomials L_n and uniform weight $w(t) = \frac{1}{b-a}$:

$$\begin{aligned} L_{n,x}(t) & := \frac{1}{n!(b-a)} \left[n(x-a - \frac{(b-a)^3}{6(2x-a-b)^2})(t-x)^{n-1} \cdot 1_{\{n \geq 1\}} \right. \\ & + \binom{n}{2} (x-a)^2 (t-x)^{n-2} \cdot 1_{\{n \geq 2\}} + \binom{n}{3} (x-a)^3 (t-x)^{n-3} \cdot 1_{\{n \geq 3\}} \\ & \left. + \binom{n}{4} (x-a)^4 (t-x)^{n-4} \cdot 1_{\{n \geq 4\}} \right], \quad t \in [x, \frac{a+b}{2}]. \end{aligned} \tag{2.15}$$

After some calculation from Theorem 5 we get

$$A_{1, \frac{1}{b-a}}^{(3)}(x) = \frac{(b-a)^2}{6(2x-a-b)^2} \quad B_{1, \frac{1}{b-a}}^{(3)}(x) = 1 - \frac{(b-a)^2}{3(2x-a-b)^2},$$

$$A_{2, \frac{1}{b-a}}^{(3)}(x) = A_{3, \frac{1}{b-a}}^{(3)}(x) = A_{4, \frac{1}{b-a}}^{(3)}(x) = B_{3, \frac{1}{b-a}}^{(3)}(x) = 0.$$

Inserting polynomials $\{L_{n,x}\}$ in Theorem 5 and Theorem 6 we get:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t)dt - D(f, x) - T_{n, \frac{1}{b-a}}^{(3)}(x) \right| \\ & \leq C_3(n, p, x, \frac{1}{b-a}) \cdot \|f^{(n)}\|_p, \end{aligned} \tag{2.16}$$

where

$$D(f, x) = \frac{(b-a)^2}{6(2x-a-b)^2} (f(x) + f(a+b-x)) + \left(1 - \frac{(b-a)^2}{3(2x-a-b)^2}\right) f\left(\frac{a+b}{2}\right), \tag{2.17}$$

$$\begin{aligned} T_{n, \frac{1}{b-a}}(f, x) &= \sum_{k=5}^n A_{k, \frac{1}{b-a}}^{(3)}(x) \left(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a+b-x) \right) \\ &+ \sum_{\substack{k=5 \\ \text{odd}}}^n B_{k, \frac{1}{b-a}}^{(3)}(x) f^{(k-1)}\left(\frac{a+b}{2}\right) \end{aligned} \tag{2.18}$$

and

$$C_3(n, p, x, \frac{1}{b-a}) = \begin{cases} 2^{1/q} \left[\int_a^{\frac{a+b}{2}} \left| W_{n, \frac{1}{b-a}}^{(3)}(t, x) \right|^q dt \right]^{1/q}, & \frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p \leq \infty \\ \sup \{ |W_{n,w}^{(3)}(t, x)|, t \in [a, \frac{a+b}{2}] \}, & p = 1, \end{cases} \tag{2.19}$$

3. OSTROWSKI-GRÜSS TYPE INEQUALITIES FOR THE REMAINDER OF THE WEIGHT THREEPOINT FORMULA

Let us apply identity (1.1) for $f \leftrightarrow (-1)^n W_{n,w}(\cdot, x)$ and $g \leftrightarrow f^{(n)}$:

$$\begin{aligned} T((-1)^n W_{n,w}^{(3)}(\cdot, x), f^{(n)}) &= \frac{1}{b-a} \int_a^b (-1)^n W_{n,w}^{(3)}(t, x) f^{(n)}(t) dt \\ &- \frac{1}{b-a} \int_a^b (-1)^n W_{n,w}^{(3)}(t, x) dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t) dt. \end{aligned} \tag{3.1}$$

Theorem 7. *Let $f : [a, b] \rightarrow \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous function, $w : [a, b] \rightarrow [0, \infty)$ is weight function such that $w(t) = w(a+b-t)$, for each $t \in [a, b]$ and $x \in [a, \frac{a+b}{2}]$. Then the following identity holds:*

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(t)w(t)dt - \frac{1}{b-a} \left[A_{1,w}^{(3)}(x) [f(x) + f(a+b-x)] + B_{1,w}^{(3)}(x) f\left(\frac{a+b}{2}\right) + T_{n,w}^{(3)}(x) \right] \\ & - \frac{(1 + (-1)^n) A_{n,w}^{(3)}(x) + B_{n,w}^{(2)}(x)}{(b-a)^2} \left(f^{(n-1)}(b) - f^{(n-1)}(a) \right) = T((-1)^n W_{n,w}^{(3)}(\cdot, x), f^{(n)}). \end{aligned} \tag{3.2}$$

Proof. Divide relation (2.9) by $b - a$ and add to both sides of new identity term

$$\begin{aligned}
 & - \frac{1}{b-a} \int_a^b (-1)^n W_{n,w}^{(3)}(t, x) dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t) dt = \\
 & - \frac{(1 + (-1)^n) A_{n,w}^{(3)}(x) + B_{n,w}^{(3)}(x)}{(b-a)^2} \left(f^{(n-1)}(b) - f^{(n-1)}(a) \right).
 \end{aligned}$$

Now the right side of the identity appears to be Chebyshev functional $T((-1)^n W_{n,w}^{(3)}(\cdot, x), f^{(n)})$ so the theorem is proved. □

Now we establish error bounds for $T((-1)^n W_{n,w}^{(3)}(\cdot, x), f^{(n)})$.

Theorem 8. *Let $f : [a, b] \rightarrow \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous function with*

$$(\cdot - a)(b - \cdot) \left(f^{(n+1)} \right)^2 \in L[a, b],$$

$w : [a, b] \rightarrow [0, \infty)$ is weight function such that $w(t) = w(a+b-t)$, for each $t \in [a, b]$ and $x \in [a, \frac{a+b}{2}]$. Then we have

$$\begin{aligned}
 \left| T((-1)^n W_{n,w}^{(3)}(\cdot, x), f^{(n)}) \right| & \leq \frac{1}{b-a} \left[(C_3(n, 2, x, w))^2 - \frac{\left((1 + (-1)^{n+1}) A_{n+1,w}^{(3)}(x) + B_{n+1,x}^{(3)}(x) \right)^2}{b-a} \right]^{1/2} \\
 & \cdot \left[\frac{1}{2} \int_a^b (t-a)(b-t) f^{(n+1)}(t)^2 dt \right]^{1/2}. \tag{3.3}
 \end{aligned}$$

Proof. The proof follows from Cauchy-Schwartz's inequality

$$T(\varphi, \psi)^2 \leq T(\varphi, \varphi) \cdot T(\psi, \psi)$$

applied to $T((-1)^n W_{n,w}^{(3)}(\cdot, x), f^{(n)})$. So we have

$$\left| T((-1)^n W_{n,w}^{(3)}(\cdot, x), f^{(n)}) \right| \leq \sqrt{T((-1)^n W_{n,w}^{(3)}(\cdot, x), (-1)^n W_{n,w}^{(3)}(\cdot, x))} \cdot \sqrt{T(f^{(n)}, f^{(n)})}. \tag{3.4}$$

We compute

$$\begin{aligned}
 T((-1)^n W_{n,w}^{(3)}(\cdot, x), (-1)^n W_{n,w}^{(3)}(\cdot, x)) & = \frac{1}{b-a} \int_a^b (-1)^{2n} W_{n,w}^{(3)}(t, x)^2 dt \\
 & - \frac{1}{(b-a)^2} \left[\int_a^b (-1)^n W_{n,w}^{(3)}(t, x) dt \right]^2 \\
 & = \frac{[C_3(n, 2, x, w)]^2}{b-a} - \frac{\left[(1 + (-1)^{n+1}) A_{n+1,w}^{(3)}(x) + B_{n+1,x}^{(3)}(x) \right]^2}{(b-a)^2}.
 \end{aligned}$$

On the other side, according to the Lemma 1. we have

$$T(f^{(n)}, f^{(n)}) \leq \frac{1}{2(b-a)} \int_a^b (t-a)(b-t) f^{(n+1)}(t)^2 dt,$$

which finishes the proof. □

Corollary 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be such that f' is absolutely continuous function with $(\cdot - a)(b - \cdot)(f'')^2 \in L[a, b]$ and $x \in [a, \frac{a+b}{2}]$. Then we have

$$\left| T(-W_{1, \frac{1}{b-a}}^{(3)}(\cdot, x), f') \right| \leq \frac{1}{b-a} \left[\left(C_3 \left(1, 2, x, \frac{1}{b-a} \right) \right)^2 - \frac{\left(2A_{2, \frac{1}{b-a}}^{(3)}(x) + B_{2, \frac{1}{b-a}}^{(3)}(x) \right)^2}{b-a} \right]^{1/2} \cdot \left[\frac{1}{2} \int_a^b (t-a)(b-t) f''(t)^2 dt \right]^{1/2}. \quad (3.5)$$

Proof. We apply Theorem 8 for $w(t) = \frac{1}{b-a}$ and $n = 1$. \square

Remark 4. Inserting harmonic polynomials $L_{n,w}$ in (3.5) we get:

$$\begin{aligned} & \left| T(-W_{1, \frac{1}{b-a}}^{(3)}(\cdot, x), f') \right| \leq \\ & \leq \frac{1}{b-a} \sqrt{\frac{b-a}{12} \left(1 - \frac{(b-a)^2}{(2x-a-b)^2} + \frac{4(x-a)^2}{(2x-a-b)^2} - \frac{(b-a)^3}{3(2x-a-b)^3} \right)} \\ & \cdot \left[\frac{1}{2} \int_a^b (t-a)(b-t) f''(t)^2 dt \right]^{1/2}. \end{aligned}$$

Specially, for $x = a$ we have the inequality related to of the Simpson quadrature formula:

$$\left| T(-W_{1, \frac{1}{b-a}}^{(3)}(\cdot, a), f') \right| \leq \frac{1}{6} \left[\frac{1}{2} \int_a^b (t-a)(b-t) f''(t)^2 dt \right]^{1/2}.$$

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FACULTY OF GEOTECHNICAL ENGINEERING, UNIVERSITY OF ZAGREB, HALLEROVA ALEJA 7, 42 000 VARAŽDIN, CROATIA.

Email address: sanja.kovac@gfv.unizg.hr

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$$1) \int \frac{\sqrt{x} dx}{(a \pm bx)^{m-1}}$$

$$\int \frac{x\sqrt{x} dx}{a - bx} = \frac{6a\sqrt{x} - 2bx}{3b^2}$$

$$\frac{a - x + x\sqrt{x}}{(a - bx)^{m-1}} + \frac{3}{2(m-1)}$$

$$= \frac{2a\sqrt{x} + \frac{a\sqrt{a}}{b^2\sqrt{b}} \ln \left| \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}} \right|}{2(m-1)}$$