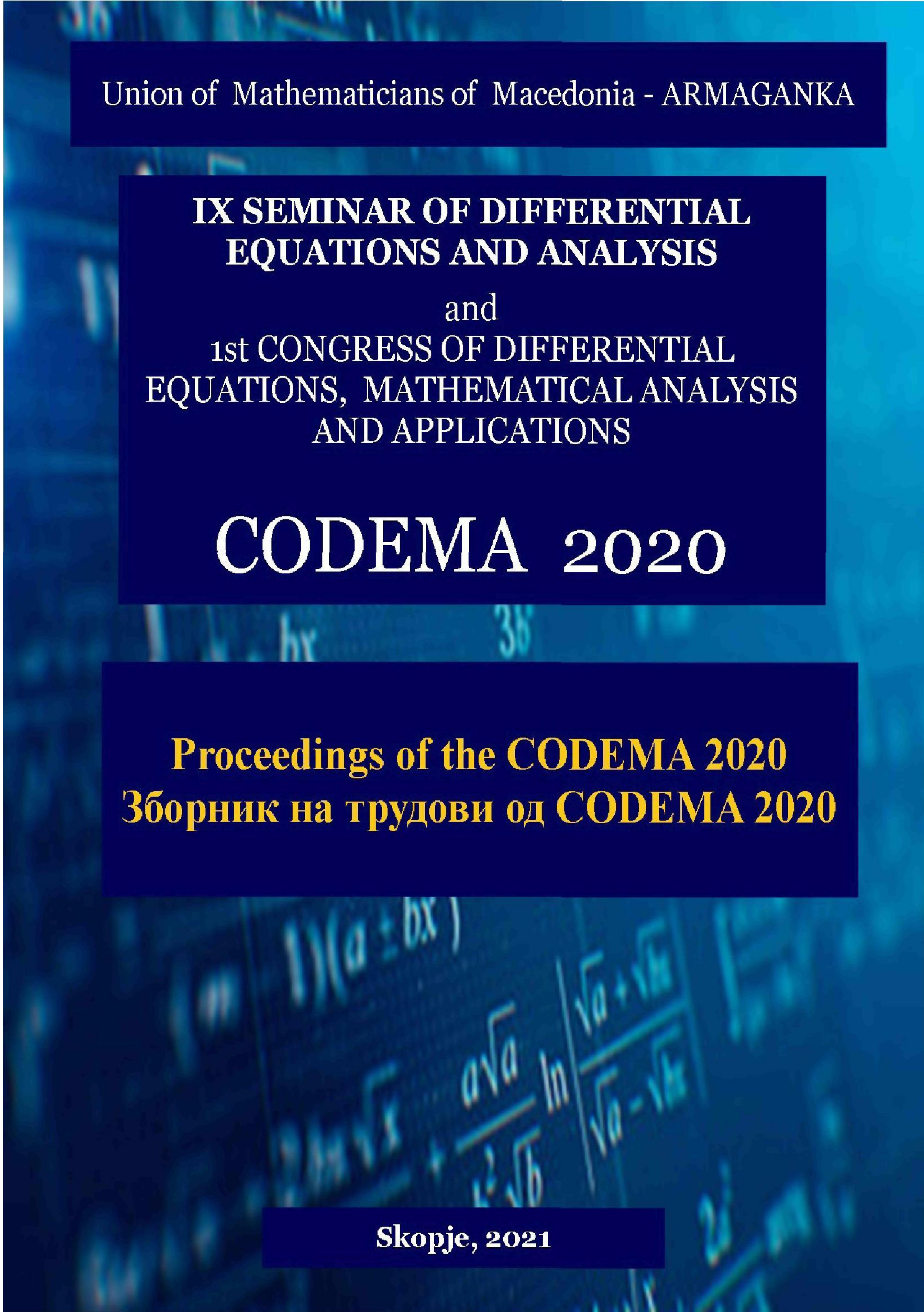

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EXTENSION OF A CYCLIC 2-SUBSPACE GENERATED BY FOUR 2-VECTORS AND SOME EXTENSIONS OF HAHN-BANACH TYPE FOR SKEW-SYMMETRIC 2-LINEAR FUNCTIONALS DEFINED ON IT

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Abstract. In this paper 2-subspaces from 2-space X^2, which are from the type - cyclic 2-subspace generated by four 2-vectors, will be taken in consideration. Then all its possible extensions adding one element (u,v) and their complete description will be considered. Also, all extensions of 2-skew-symmetric linear form defined on 2-subspace M' Hahn-Banach type will be considered, in the cases when one vector belongs in 2-vector from M, and the other does not belong in any 2-vector, as well as cases when the two coordinates of (u,v) do not belong in M.

1. INTRODUCTION

Extensions of mappings is something that is often looked at in various mathematical disciplines. One classical example of extension of a given mapping is of course the Hanh-Banach theorem for linear functionals. One version of it comprises the contents of the following theorem.

Theorem 1. *Let* M *be a vector subspace of the vector space* X. *The functional* $p: X \to \mathbb{R}$ *satisfies the conditions*

a) $p(x+y) \leq p(x) + p(y)$

b) $p(tx) = tp(x)$,

for every $x, y \in X$ *and* $t \geq 0$.

The functional $f: M \to R$ *is linear and* $f(x) \leq p(x)$. *There exists a linear functional* $\Lambda: X \to \mathbb{R}$ *such that* $\Lambda / M = f$ *and* $-p(-x) \leq \Lambda(x) \leq p(x)$.

Due to the definition of an n-norm and the definition of an n-semi norm it turned out that, on the set X^2, where X is a vector space over the field Φ (Φ is the field of real numbers or the field of complex numbers), it is convenient to consider additional operations, two of which are partial and one of which is a complete operation, with the aim of making the notation and considerations easier.

Of course, it is worth mentioning here both the definitions, for 2-norm, and especially for 2 semi-norm, which we will use many times further.

Definition 1. Let X be a vector space over the field Φ. The mapping $\|\bullet,\bullet\|: X^2 \to \mathbb{R}_{\geq 0}$ for which the following conditions are fulfilled

(i) $\|x,y\| = 0$ if and only if $\{x,y\}$ is a linear dependent set

(ii) $\|x,y\| = \|y,x\|$ for arbitrary $x, y \in X$

(iii) $\|\alpha x, y\| = |\alpha| \cdot \|x,y\|$ for arbitrary $\alpha \in \Phi$ and for arbitrary $x, y \in X$

(iv) $\|x+x', y\| \leq \|x,y\| + \|x',y\|$, for arbitrary $x, y \in X$,

we call **2-norm**, and $(X^2, \|\bullet,\bullet\|)$ we call **2-normed space**.

Definition 2. Let X is a vector space over the field Φ. The mapping $p: X^2 \to \mathbb{R}_{\geq 0}$ for which the following conditions are fulfilled

(i) $p(x,y) \geq 0$ if and only if $\{x,y\}$ is a linear dependent set

(ii) $p(x,y) = p(y,x)$ for arbitrary $x, y \in X$

(iii) $p(\alpha x, y) = |\alpha| \cdot p(x,y)$ for arbitrary $\alpha \in \Phi$ and for arbitrary $x, y \in X$

(iv) $p(x+x', y) \leq p(x,y) + p(x',y)$, for arbitrary $x, y \in X$,

we call **2-semi norm**, and (X^2, p) we call **2-semi normed space**.

It is worth to note here that for any 2-norm the following equation is fulfilled $\|x,y\| = \|x, y+\alpha x\|$, for arbitrary $x, y \in X$ and for arbitrary scalar $\alpha \in \Phi$.

With further consideration and development of the previous equation, the following definitions for 2-semi-normed space, for 2-norm and for 2-semi-norm were directly imposed.

Definition 3. Let X be a vector space over the field Φ. The mapping $p: X^2 \to \mathbb{R}$ for which the following conditions hold

(a) $p(x+y, z) \leq p(x,z) + p(y,z)$, for every $x, y, z \in X$

(b) $p(A(x,y)) = |\det A|\, p(x,y)$, for every $x, y \in X$ and $A \in M_2(\Phi)$.

is called a **2-semi norm** and (X^2, p) is called a **2-semi normed space**.

Definition 4. The mapping $\|\cdot\|: X^n \to \mathbb{R}$, $n \geq 2$ for which it is fulfilled that:

(a) $\|x_1, x_2\| = 0$ if and only if x_1, x_2 are linear dependant vectors;

(b) $\|A(x_1, x_2)\| = |\det A|\, \|x_1, x_2\|$, for all $x_1, x_2 \in X$ and for all $A \in M_2(\Phi)$;

(c) $\|x_1 + x_2, x_3\| \leq \|x_1, x_3\| + \|x_2, x_3\|$, for all $x_1, x_2, x_3 \in X$,

we call **2-norm** of the vector space X, and the ordered pair $(X, \|\cdot,\cdot\|)$ we call **2-normed space**.

Of course this in itself has led to consideration of definitions for addition operations and 2-vector multiplication operations with scalar, which are the basic operations in X^2. In other words, we will consider, i.e. under consideration is the following definition.

Definition 5. Let X be a vector space over the field Φ. The set X^2 together with the operations

$$(x,z) + (y,z) = (x+y, z)$$

$$(z,x) + (z,y) = (z, x+y)$$

$$A(x,y) = A(x,y)^T$$

where $x, y, z \in X$ and $A \in M_2(\Phi)$ is called a **2-vector space** or **2-space**.

Comment. The third operation in the previous definition is a complete operation, and on the right-hand side of the equality is a multiplication of a matrix with a vector.

Definition 6. Let X be a vector space over the field Φ. The functional $\Lambda: X^2 \to \Phi$ for which the following conditions hold *(a)* $\Lambda(x+y,z)=\Lambda(x,z)+\Lambda(y,z)$, for arbitrary $x,y,z\in X$

(b) $\Lambda(x,y)=-\Lambda(y,x)$ for arbitrary $x,y\in X$

(c) $\Lambda(\alpha x,y)=\alpha\Lambda(x,y)$, for arbitrary $x,y\in X$ and $\alpha\in\Phi$.

is called **skew-symmetric 2-linear form**.

It is not hard to prove that the previous definition (Definition 6) is equivalent with the following definition.

Definition 7. Let X be a vector space over the field Φ. The functional $\Lambda:X^2\to\Phi$ for which the following conditions hold

(a) $\Lambda(x+y,z)=\Lambda(x,z)+\Lambda(y,z)$, for arbitrary $x,y,z\in X$

(b) $\Lambda(A(x,y))=(\det A)\Lambda(x,y)$, for arbitrary $x,y\in X$ and $A\in M_2(\Phi)$

is called **skew-symmetric 2-linear form** or simply **2-linear functional**.

This makes levelling to all previous definitions for such considerations.

The very definition of a 2-semi-norm and the numerous examples that were constructed after that led to a situation, analogous as in vector space, to consider different subsets. Among them are of course the sets that are closed in relation to the addition and multiplication operations with matrix as basic operations in X^2.

Definition 8. The subset S, $S\subseteq X^2$ which is closed with respect to the operations of the 2-space X^2 is called **2-subspace** of X^2.

Comment. Of course, one of the most bitter problems associated with the operations on X^2 and the subsets of X^2 at the given moment is the complete description of the structure of the 2-subspaces of the 2-space X^2. Due to this, we will focus our attention on only one special type of 2-subspaces of the 2-vector space X^2.

In the following considerations the following theorem about subspaces is useful.

Theorem 2. *The intersection of an arbitrary family of 2-subspaces of the 2-vector space X^2 is a 2-subspace.*

According to the last theorem, each subset $A\subseteq X^2$ determines a 2-subspace S_A, the smallest 2-subspace of the 2-vector space X^2 which contains the set A. We will call the 2-subspace S_A the 2-subspace **generated by the set** A, and the set A **-the generating set**.

In this matter we will consider a special type of generating sets, i.e. a generating set of the form $M\cup\{(u,v)\}$, where M is a special type of a 2-subspace, and $(u,v)\in X^2$ is arbitrarily given where $\{u,v\}$ is a linearly independent set.

The basic question which we will consider here is whether it is possible to extend a 2-skew-symmetric linear form defined on some types, i.e. classes 2-subspaces to a bigger subspace, in the sense of extension of linear functionals, i.e. of the type of Hanh-Banach.

The main aim if all such considerations is whether we can prove the following theorem or some of its variants.

Theorem 3. *Let* S *be a 2-subspace of the 2-space* X^2, $\Lambda : S \to \mathbb{R}$ *be 2-skew-symmetric linear form, and* $p : X^2 \to \mathbb{R}$ *be a mapping for which*

(a) $p(x+y,z) \le p(x,z)+p(y,z)$, *for all* $x,y,z \in X$

(b) $p(tx,y) = tp(x,y)$, *for all* $x,y \in X$ *and* $t > 0$.

There exists 2-skew-symmetric linear form $\Lambda' : X^2 \to \mathbb{R}$ *, such that* $\Lambda' / S = \Lambda$.

Each 2-seminorm satisfies the conditions a) and b) from the previous theorem.

Furthermore, in many parts we may come across a special kind of subset of X^2. One type of them is given in the following definition.

Definition 9. The subset $T, T \subseteq X^2$ is called n**-invariant** if $AT \subseteq T$ for every $A \in M_2(\Phi)$, $\det A = 1$.

Solving the problem presented in the last theorem is of course not simple. An affirmation of that is of course the complex structure of the 2-subspaces of the 2-space X^2. Due to this, we will discuss partial cases of this problem.

In this matter we will look at extension of a 2-skew-symmetric form defined on a cyclic 2-subspace.

From here on, we will assume that the subset $\{x_1, x_2, ..., x_n\}$ is a linearly independent subset of the vector space X.

Definition 10. Let X be a vector space over the field Φ. The 2-subspace S generated by the set $\{(x_1,x_2),(x_2,x_3),...,(x_n,x_1)\}, n \ge 3$, is called a **cyclic 2-subspace.**

In the next part we will deal with 2-subspaces that are generated with 4 elements. They are quite characteristic and in many ways significantly different from other cyclic 2-subspaces that are generated with 5 or more elements. Therefore, definition 10 will get the following form

Definition 10'. Let X be a vector space over the field Φ. The 2-subspace S generated by the set $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),(x_4,x_1)\}$, is called a **cyclic 2-subspace generated with four elements.**

A detailed description of this kind of a 2-subspace is given in [7]. That is the content of the theorem that follows.

Theorem 4. *The cyclic 2-subspace generated by the elements of the set* $\{(x_1,x_2),(x_2,x_3),...,(x_n,x_1)\}, n \ge 5$ *is*

$$M = \bigcup_{i=1}^{n} \left[L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \right],$$

where $x_{n+1} \equiv x_1$ *and* $x_0 \equiv x_n$.

In the case $n = 4$ this theorem has the following form

Theorem 4'. *The cyclic 2-subspace generated by the elements of the set* $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),(x_4,x_1)\}$ *is*

$$M = \bigcup_{\substack{i=1 \\ x_o = x_4, x_5 = x_1}}^{4} \left[L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \right].$$

The leading result in the description of the special 2-subspaces such as cyclic, branch 2-subspaces, kernel 2-subspaces and loop 2-subspaces is the following lemma:

Lemma. *The subspace generated by the elements* $(x_{i-1}, x_i), (x_i, x_{i+1}), (x_{i+1}, x_{i+2})$, *where* $\{x_{i-1}, x_i, x_{i+1}, x_{i+2}\}$ *is a linearly independent set is*

$$L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \times L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \cup L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i)$$

The idea for such lemma is because here it seems as if we have put together two branches, i.e.

$$L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \times L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \qquad (1)$$

and $L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i)$. (2)

Here, as its 2-subspace appears a set determined with

$$M = \{(A(x_i, x_{i+1})^T \,/\, A \in M_2(\Phi)\}.$$

Addition of elements from (1) and (2) certainly is possible, but the result is always an element which can be considered that belongs in one of these 2-subspaces, i.e. either in (1) or in (2), except maybe in some special cases when it belongs in the 2-subspace $M = \{(A(x_i, x_{i+1})^T \,/\, A \in M_2(\Phi)\}$.

2. EXTENSION OF A CYCLIC 2-SUBSPACE

Let Λ be a skew-symmetric linear form defined on a branch 2-subspace M, generated by the elements of the set $\{(x_1, x_2), (x_2, x_3), (x_3, x_4), \ldots, (x_{n-1}, x_n), (x_n, x_1)\}$ where $\{x_1, x_2, \ldots, x_n\}$ is a linearly independent set. Let $(u, v) \in X^2$ be such that $\{u, v\}$ is a linearly independent set. We denote the 2-subspace of X^2 generated by $M \cup \{(u, v)\}$ by M'. Several cases are possible.

Case 1. $u, v \notin L(x_1, x_2, x_3, x_4)$, where $L(x_1, x_2, x_3, x_4)$ is the subspace of X generated by $\{x_1, x_2, x_3, x_4\}$.

Case 2. $u \in L(x_1, x_2, x_3, x_4)$ and $v \notin L(x_1, x_2, x_3, x_4)$. The case $u \notin L(x_1, x_2, x_3, x_4)$ and $v \in L(x_1, x_2, x_3, x_4)$ is completely analogous due to symmetry.

Case 3. $u, v \in L(x_1, x_2, x_3, x_4)$.

In cases 2 and 3 there are several sub cases:

Sub cases of case 1

The 2-subspace that is determined here is

$$M' = M \cup \left\{ A(u, v)^T \,/\, A \in M_2(\Phi) \right\}$$

Indeed, from the fact that $u, v \notin L(x_1, x_2, x_3, x_4)$, there exist vectors y, z which are not equal to zero vector and for which

$$u = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \gamma y = x + \gamma y$$

$$v = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \delta z = w + \delta z \,\mathbf{,}$$

where $\gamma\delta \neq 0$. But then for any matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ which is nonsingular, we have the following possibilities

1° $a_{11} \neq 0$, $a_{12} = 0$

2° $a_{11} = 0$, $a_{12} \neq 0$

3° $a_{11} \neq 0$, $a_{12} \neq 0$

For 1° we have the following three additional possibilities:

a) $a_{21} \neq 0$, $a_{22} = 0$, which is not possible because in this situation we would have that $\det A = 0$, which is not possible.

b) $a_{21} = 0$, $a_{22} \neq 0$, which is possible. In this situation $\det A = a_{11}a_{22} \neq 0$. Here

$$A(u,v)^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = (a_{11}u, a_{22}v) = (a_{11}(x+\gamma y), a_{22}(w+\delta z)) = (a_{11}x + a_{11}\gamma y, a_{22}w + a_{22}\delta z),$$

where from because of the condition $a_{11}\gamma a_{22}\delta \neq 0$, we get that $A(u,v)^T \notin M$. This element will belong in the new set which is added.

c) $a_{21} \neq 0$, $a_{22} \neq 0$, which is possible. In this situation $\det A = a_{11}a_{22} \neq 0$. Here, same as the previous case

$$A(u,v)^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = (a_{11}u, a_{21}u + a_{22}v) = (a_{11}(x+\gamma y), a_{21}(x+\gamma y) + a_{22}(w+\delta z)) =$$
$$= (a_{11}x + a_{11}\gamma y, a_{21}x + a_{22}w + a_{21}\gamma y + a_{22}\delta z)$$

where from because of the condition $a_{11}\gamma \neq 0$ we have that $A(u,v)^T \notin M$. This element belongs in the new set $\{A(u,v)^T / A \in M_2(\Phi)\}$.

For 2° we have the following three additional possibilities:

a) $a_{21} \neq 0$, $a_{22} = 0$, which is possible, because in this case we have that

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = -a_{12}a_{21} \neq 0.$$

But, in this case we have that

$$A(u,v)^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = (a_{12}v, a_{21}u) = (a_{12}(w+\delta z), a_{21}(x+\gamma y)) = (a_{12}w + a_{12}\delta z, a_{21}x + a_{21}\gamma y)$$

and now because of the condition $a_{12}\delta a_{21}\gamma \neq 0$, $A(u,v)^T \notin M$.

b) $a_{21} = 0$, $a_{22} \neq 0$, which is possible from technical view. But,

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = 0 \cdot a_{22} - 0 \cdot a_{21} = 0,$$

And by starting assumption we have that $\det A \neq 0$. Because of this contradiction, this case is not possible in this situation.

c) $a_{21} \neq 0$, $a_{22} \neq 0$. This case is possible from technical view. Indeed,

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = 0 \cdot a_{22} - a_{12} \cdot a_{21} = -a_{12} \cdot a_{21} \neq 0.$$

In this case we have that

$$A(u,v)^T=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}=(a_{12}v,a_{21}u+a_{22}v)=(a_{12}(w+\delta z),a_{21}(x+\gamma y)+a_{22}(w+\delta z))=$$
$$=(a_{12}w+a_{12}\delta z,a_{21}x+a_{21}\gamma y+a_{22}w+a_{22}\delta z)=$$
$$=a_{12}(w+\delta z,a_{21}x+a_{21}\gamma y+a_{22}w+a_{22}\delta z)\notin M$$

because the first component $w+\delta z\in L(x_1,x_2,x_3,x_4)$.

For 3° we have the following three additional possibilities:

a) $a_{21}\neq 0$, $a_{22}=0$, which is possible, because in this case we have that

$$\det A=\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=-a_{21}a_{12}\neq 0,$$

and the matrix is nonsingular. According to that

$$A(u,v)^T=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}=(a_{12}v,a_{21}u)=(a_{12}(w+\delta z),a_{21}(x+\gamma y))=$$
$$=a_{12}a_{21}(w+a_{12}\delta z,x+\gamma y)\notin M$$

b) $a_{21}=0$, $a_{22}\neq 0$, which is also possible, where

$$\det A=\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}\neq 0.$$ Here,

$$A(u,v)^T=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}=(a_{11}u,a_{22}v)=(a_{11}(x+\gamma y),a_{22}(w+\delta z))=$$
$$=a_{11}a_{22}(x+\gamma y,w+a_{12}\delta z)\notin M$$

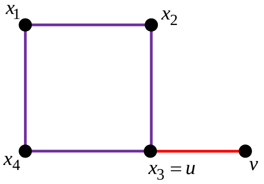
c) $a_{21}\neq 0$, $a_{22}\neq 0$. This case because of its nature is the most radical one. But, here, if we use the technique which we have in the 2-normed spaces, we have that

$$A(u,v)^T=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}=(a_{11}u+a_{12}v,a_{21}u+a_{22}v)=$$
$$=(a_{11}(x+\gamma y)+a_{12}(w+\delta z),a_{21}(x+\gamma y)+a_{22}(w+\delta z))=$$
$$=(a_{11}x+a_{11}\gamma y+a_{12}w+a_{12}\delta z,a_{21}x+a_{22}w+a_{21}\gamma y+a_{22}\delta z)\sim(=)$$
$$=(\det A)(x+\gamma y,w+\delta z)=((\det A)x+(\det A)\gamma y,w+\delta z)\notin M$$

because $(\det A)\gamma\neq 0$, where from we get that the first element does not belong in any element of M and with that the whole element does not belong in M. Also, let us note that in the part where we have $\sim(=)$ it is a sign of equality. But, that is not a problem, because from that element till the ending element we constantly multiply with a matrix that has a determinant equal to 1, so according to this if one element does not belong in M, then also any element multiplied with a matrix that has a determinant equal to 1 does not belong in M. The last equality may not be used, because $\det A\neq 0$ and $\gamma\neq 0$ where from follows the proof.

Sub cases of case 2

Sub case 1. $u = x_i$, for any $i = 1,2,3,4$, which in the discussion is a fixed element.



In this case we have a situation as given in the scheme in case when we have 4 generator 2-vectors; the rest of the cases are equal to this one that is illustrated (see the drawing; in other words, the vector u can be any of the vectors x_1, x_2, x_3, x_4; here, only the case $u = x_3$ is considered, and all other cases are equivalent to that one).

In this situation, we have the existing cyclic 2-subspace, which has undistorted structure with the addition of one element and a structure of one pure loop 2-subspace S, generated by the elements $(u,v) = (x_3,v), (u,x_2) = (x_3,x_2)$ and $(u,x_4) = (x_3,x_4)$. Let us mention here that

$$S = \bigcup_{w \in L(v,x_2,x_4)'} L(w,u) \times L(w,u) .$$

We are interested especially the cases when $w = \alpha v + \beta x_2 + \gamma x_4$, where especially $\alpha \neq 0$. Since

$$M = \bigcup_{\substack{i=1 \\ x_0 = x_4, x_5 = x_1}}^{4} \left[L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \right]$$

we can prove directly that

$$M' = M \cup S = \bigcup_{\substack{i=1 \\ x_0 = x_4, x_5 = x_1}}^{4} \left[L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \right] \cup \left[\bigcup_{w \in L(v,x_2,x_4)'} L(w,u) \times L(w,u) \right]$$

Indeed, if we have an element from the set S, it has the form $A(u, \alpha v + \beta x_2 + \gamma x_4)$ where we consider that $\det A \neq 0$, and let us have an element from the cyclic 2-subspace. Then, that element has to have the form

a) $B(u, \alpha_2 x_2 + \alpha_4 x_4)$, $B \in M_2(\Phi)$

b) $C(x_4, \alpha_3 u + \alpha_1 x_1)$, $C \in M_2(\Phi)$

c) $D(x_1, \alpha_2 x_2 + \alpha_4 x_4)$, $D \in M_2(\Phi)$

d) $E(x_2, \alpha_1 x_1 + \alpha_4 x_4)$, $E \in M_2(\Phi)$.

Everyone of this cases will be considered separately, where we will consider that $\det B, \det C, \det D, \det E \neq 0$

Case c) In this case every vector from $D(x_1, \alpha_2 x_2 + \alpha_4 x_4)$ has the form

$$D(x_1, \alpha_2 x_2 + \alpha_4 x_4) = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} (x_1, \alpha_2 x_2 + \alpha_4 x_4) =$$

$$= (d_{11}x_1 + d_{12}(\alpha_2 x_2 + \alpha_4 x_4), d_{21}x_1 + d_{22}(\alpha_2 x_2 + \alpha_4 x_4)).$$

Since $\det D \neq 0$, then either $d_{11} \neq 0$ or $d_{21} \neq 0$, so, according to this either in the first coordinate or in the second coordinate there will exist somewhere in the adding a vector x_1 and because of that cannot be equal with any element from S, which has the form

$$A(u,\alpha v+\beta x_2+\gamma x_4)=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}(u,\alpha v+\beta x_2+\gamma x_4)=$$

$$=(a_{11}x_3+a_{12}(\alpha v+\beta x_2+\gamma x_4),a_{21}x_3+a_{22}(\alpha v+\beta x_2+\gamma x_4))$$

In the 2-vector $D(x_1,\alpha_2 x_2+\alpha_4 x_4)$ in one of the coordinates always there will appear x_1. So, according to this, $S\cap M=\varnothing$ (or $S\cap M\subseteq\Delta_2$) in this situation.

Case b) In this case, we have that the elements from $C(x_4,\alpha_3 u+\alpha_1 x_1)$ has the form

$$C(x_4,\alpha_3 u+\alpha_1 x_1)=C(x_4,\alpha_3 x_3+\alpha_1 x_1)=\begin{bmatrix} c_{11} & c_{12} \\ c_{13} & c_{14} \end{bmatrix}(x_4,\alpha_3 x_3+\alpha_1 x_1)=$$

$$=(c_{11}x_4+c_{12}(\alpha_3 x_3+\alpha_1 x_1),c_{21}x_4+c_{22}(\alpha_3 x_3+\alpha_1 x_1))=(\sim)$$

$$=(\det C)(x_4,\alpha_3 x_3+\alpha_1 x_1)=$$

But, the matrix C is nonsingular, so according to this either $c_{11}\neq 0$ or $c_{12}\neq 0$, so in order to have operation addition, it has to $c_{12}\neq 0$, and $c_{11}=0$. In this case we have that α_1 has to be equal to zero, $a_{12}=0$, where $c_{12}\alpha_3=a_{11}$. Then, we would have that

$$A(u,\alpha v+\beta x_2+\gamma x_4)=\begin{bmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}(u,\alpha v+\beta x_2+\gamma x_4)=(a_{12}x_3,a_{21}x_3+a_{22}(\alpha v+\beta x_2+\gamma x_4))$$

$$C(x_4,\alpha_3 u+\alpha_1 x_1)=\begin{bmatrix} 0 & c_{12} \\ c_{21} & c_{22} \end{bmatrix}(x_4,\alpha_3 x_3)=(c_{12}\alpha_3 x_3,c_{21}x_4+c_{22}(\alpha_3 x_3+\alpha_1 x_1))$$

and the addition can be done. The result is always a 2-vector is both in S and in M , as well.

Case d) This case due to the complete symmetry is fully analogous to the previous Case b) and there is no need to be considered.

Case a) It is clear that this case is absolutely possible. The general member from the subspace generated from the element $(u,\alpha_2 x_2+\alpha_4 x_4)=(x_3,\alpha_2 x_2+\alpha_4 x_4)$ is equal to

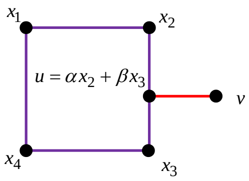
$$B(u,\alpha_2 x_2+\alpha_4 x_4)=(b_{11}x_3+b_{12}(\alpha_2 x_2+\alpha_4 x_4),b_{21}x_3+b_{22}(\alpha_2 x_2+\alpha_4 x_4))\ .$$

It is clear that he elements from the 2-subspace generated from $(u,\alpha v+\beta x_2+\gamma x_4)=(x_3,\alpha v+\beta x_2+\gamma x_4)$, have the following form

$$A(u,\alpha v+\beta x_2+\gamma x_4)=A(x_3,\alpha v+\beta x_2+\gamma x_4)=$$

$$=(a_{11}x_3+a_{12}(\alpha v+\beta x_2+\gamma x_4),a_{21}x_3+a_{22}(\alpha v+\beta x_2+\gamma x_4))$$

It is obvious that addition is possible in more than one variant and the result is always an element from the new 2-subspace described before.



Sub case 2. $u=\alpha x_i+\beta x_{i+1}$ where $\alpha,\beta\neq 0$

In this situation, the element u can belong in any 2-dimensional vector subspace $L(x_i,x_{i+1})$ for $u=1,2,3,4$

where $x_5 \equiv x_1$. Because of the previous considerations we have that $u = \alpha x_2 + \beta x_3, \alpha\beta \neq 0$.

Here, because of determination, we choose $u \in L(x_2, x_3)$, i.e. $u = \alpha x_2 + \beta x_3, \alpha\beta \neq 0$ (because of the condition $\alpha\beta \neq 0$ we have that $\alpha \neq 0$ and $\beta \neq 0$; it means that $u \neq x_2$ and $u \neq x_3$; so, u is any element, which is fixed, from the 2-dimensional vector subspace from the vector space X except the vectors x_2 and x_3, but the vector u is a fixed vector. Completely analogous in total, are considered the following three cases:

a) $u \in L(x_1, x_2)$, i.e. $u = \alpha x_1 + \beta x_2$, $\alpha\beta \neq 0$

b) $u \in L(x_3, x_4)$, i.e. $u = \alpha x_3 + \beta x_4$, $\alpha\beta \neq 0$

c) $u \in L(x_4, x_1)$, i.e. $u = \alpha x_4 + \beta x_1$, $\alpha\beta \neq 0$

In this situation which we will consider, we have a situation in which $(u,v),(u,x_2)$ forms one branch , from one side, and $(u,v),(u,x_3)$ another branch, from the other side, where $u = \alpha x_2 + \beta x_3$, $\alpha\beta \neq 0$.

Can we consider, in this case, the vector $u = \alpha x_2 + \beta x_3$ as a loop in one loop 2-subspace which is generated by the three 2-vectors $(u,v),(u,x_2),(u,x_3)$ where $u = \alpha x_2 + \beta x_3, \alpha\beta \neq 0$? It is clear that u is generated from two vectors which build the starting cyclic 2-subspace, i.e.

$$(u,m) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} (x_2, x_3)^T = (\alpha x_2 + \beta x_3, \gamma x_2 + \delta x_3) .$$

Now, from formal point of view we have that:

- All elements from the 2-subspace generated by $(u,v),(u,x_2)$ are 2-vectors in the following form

$$A(\alpha_1 v + \alpha_2 x_2, u)^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} (\alpha_1 v + \alpha_2 x_2, \alpha x_2 + \beta x_3)^T =$$
$$= (a_{11}(\alpha_1 v + \alpha_2 x_2) + a_{12}(\alpha x_2 + \beta x_3), a_{21}(\alpha_1 v + \alpha_2 x_2) + a_{22}(\alpha x_2 + \beta x_3)) =$$
$$= (a_{11}\alpha_1 v + (a_{11}\alpha_2 + a_{12}\alpha) x_2 + a_{12}\beta x_3, a_{21}\alpha_1 v + (a_{21}\alpha_2 + a_{22}\alpha) x_2 + a_{22}\beta x_3)$$

All elements from the 2-subspace generated by $(u,v),(u,x_3)$ are 2-vectors in the following form

$$B(\beta_1 v + \beta_3 x_3, u)^T = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} (\beta_1 v + \beta_3 x_3, \alpha x_2 + \beta x_3)^T =$$
$$= (b_{11}(\beta_1 v + \beta_3 x_3) + b_{12}(\alpha x_2 + \beta x_3), b_{21}(\beta_1 v + \beta_3 x_3) + b_{22}(\alpha x_2 + \beta x_3)) =$$
$$= (b_{11}\beta_1 v + b_{12}\alpha x_2 + (b_{11}\beta_3 + b_{12}\beta) x_3, b_{21}\beta_1 v + b_{22}\alpha x_2 + (b_{21}\beta_3 + b_{22}\beta) x_3)$$

These are the two most general elements from the two subspaces and over them addition can be done only in different cases, which will be considered further in the text. Here in order addition to be possible, one of the coordinates has to be the same. We will choose only the first coordinate to be the same. The considerations for the second coordinate is the same as for the first one and in given moment we may include it too, parallel to the first one, if there is a need for it. Here we have the equation for the first two coordinates which has to be fulfilled:

$$a_{11}(\alpha_1 v+\alpha_2 x_2)+a_{12}(\alpha x_2+\beta x_3)=b_{11}(\beta_1 v+\beta_3 x_3)+b_{12}(\alpha x_2+\beta x_3) \qquad (*)$$

i.e. $a_{11}\alpha_1 v+(a_{11}\alpha_2+a_{12}\alpha)x_2+a_{12}\beta x_3=b_{11}\beta_1 v+b_{12}\alpha x_2+(b_{11}\beta_3+b_{12}\beta)x_3$, (**) having in mind that $\alpha\beta\neq 0$, i.e. $\alpha,\beta\neq 0$.

We will consider all possible cases for equality of the two vectors by the first coordinate. Here we get the system of equations

$$\begin{cases} a_{11}\alpha_1=b_{11}\beta_1 \\ a_{11}\alpha_2+a_{12}\alpha=b_{12}\alpha \\ a_{12}\beta=b_{11}\beta_3+b_{12}\beta \end{cases}.$$

We will determine when this system has solutions.

Case 1. $a_{11}=0$.

Then the system gets the form

$$\begin{cases} b_{11}\beta_1=0 \\ a_{12}\alpha=b_{12}\alpha \\ a_{12}\beta=b_{11}\beta_3+b_{12}\beta \end{cases} \quad \text{and because } \alpha,\beta\neq 0 \text{ we get the system } \begin{cases} b_{11}\beta_1=0 \\ a_{12}=b_{12} \\ b_{11}\beta_3=0 \end{cases}$$

Then (*) gets the form

$$a_{12}(\alpha x_2+\beta x_3)=b_{11}(\beta_1 v+\beta_3 x_3)+b_{12}(\alpha x_2+\beta x_3) \qquad (*),$$

i.e. $b_{11}(\beta_1 v+\beta_3 x_3)=0$.

Sub case 1. Now, if $b_{11}=0$, then for the first coordinate we have that $a_{12}(\alpha x_2+\beta x_3)=b_{12}(\alpha x_2+\beta x_3)$ and addition is possible and the sum is clear.

Sub case 2. If $b_{11}\neq 0$, then the two scalars β_1 and β_3 are zero. But then practically we get an element from the set Δ_2, i.e. $B(\beta_1 v+\beta_3 x_3,u)^T=B(o,u)^T\in\Delta_2$. Now, because $a_{12}=b_{12}$ addition is possible and the sum is clear.

Case 2. $a_{11}\neq 0$.

Then, the system gets the form

$$\begin{cases} a_{11}\alpha_1=b_{11}\beta_1 \\ a_{11}\alpha_2=(b_{12}-a_{12})\alpha \\ b_{11}\beta_3=(a_{12}-b_{12})\beta \end{cases}$$

Sub case 1. In this situation it is possible $b_{11}=0$, but then because of the condition $\beta\neq 0$, we would have that $a_{12}=b_{12}$. Then, for the first coordinate of the 2-vectors we would have that the equality $a_{12}(\alpha x_2+\beta x_3)=b_{12}(\alpha x_2+\beta x_3)$ holds.

So, addition is possible and here we would have a 2-vector which belongs same as when the vector u is a loop element.

Sub case 2. Also, in this situation it is possible that $b_{11}\neq 0$, too. But, then we would have a system which will come down to the following

$$\begin{cases} \alpha_1 = \dfrac{b_{11}\beta_1}{a_{11}} \\ \dfrac{a_{11}\alpha_2}{\alpha} = b_{12} - a_{12} \\ \dfrac{b_{11}\beta_3}{\beta} = a_{12} - b_{12} \end{cases}$$

In such a situation we would have that $\alpha_1 = \frac{b_{11}\beta_1}{a_{11}}$, $a_{12} = \frac{b_{11}\beta_3}{\beta} + b_{12}$ and because of the equality $\frac{a_{11}\alpha_2}{\alpha} = -\frac{b_{11}\beta_3}{\beta}$, we get that $\alpha_2 = -\frac{b_{11}}{a_{11}}\frac{\alpha}{\beta}\beta_3$. Now, if we make a substitution in (*) or in (**) we would get that

$$\begin{aligned} a_{11}(\alpha_1 v + \alpha_2 x_2) + a_{12}(\alpha x_2 + \beta x_3) &= a_{11}\left(\frac{b_{11}}{a_{11}}\beta_1 v - \frac{b_{11}}{a_{11}}\frac{\alpha}{\beta}\beta_3 x_2\right) + a_{12}(\alpha x_2 + \beta x_3) = \\ &= b_{11}\beta_1 v - b_{11}\frac{\alpha}{\beta}\beta_3 x_2 + \left(b_{12} + \frac{b_{11}\alpha_3}{\beta}\right)(\alpha x_2 + \beta x_3) \\ &= b_{11}\beta_1 v - b_{11}\frac{\alpha}{\beta}\beta_3 x_2 + b_{12}(\alpha x_2 + \beta x_3) + b_{11}\frac{\alpha}{\beta}\beta_3 x_2 + b_{11}\alpha_3 x_3 = \\ &= b_{11}\beta_1 v + b_{11}\alpha_3 x_3 + b_{12}(\alpha x_2 + \beta x_3) = \\ &= b_{11}(\beta_1 v + \alpha_3 x_3) + b_{12}(\alpha x_2 + \beta x_3) \end{aligned}$$

According to that, the 2-vectors are additive. Their sum is not hard to calculate.

From these reasons, from formal point of view, regarding the addition of the elements from X^2, the element u will be called as a **loop/sub loop element**.

So, finally we have that

$$M' = M \cup S,$$

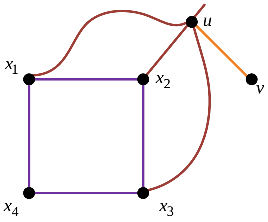
where S is that loop-sub loop 2-subspace with loop u.

Sub case 3. The vector u is a coordinate of some vector $(u,w) \in L(\alpha_1 x_1 + \alpha_3 x_3, x_2) \times L(\alpha_1 x_1 + \alpha_3 x_3, x_2)$ and $v \notin L(x_1, x_2, x_3, x_4)$, additional restrictions for u.

In this situation we have only one case which can be seen in the drawing. In this situation the vector u can be shown in the following form

$$(u,w) = A(\alpha_1 x_1 + \alpha_3 x_3, x_2)^T = (\underbrace{a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12}x_2}_{u}, \underbrace{a_{21}(\alpha_1 x_1 + \alpha_3 x_3) + a_{22}x_2}_{w}),$$

i.e. $u = a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12}x_2$, where $a_{11}, \alpha_1, \alpha_3, a_{12}$ are arbitrary scalars which at this moment are fixed. All elements which would be considered are from this type. Now, the 2-vector (u,v) has the following form $(a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12}x_2, v)$ and the elements from this 2-subspace which is generated from this 2-vector have the following form $B(a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12}x_2, v)^T$. So, their general form would be



$$B(a_{11}(\alpha_1 x_1+\alpha_3 x_3)+a_{12}x_2,v)^T=(b_{11}((\alpha_1 x_1+\alpha_3 x_3)+a_{12}x_2)+b_{12}v,b_{21}((\alpha_1 x_1+\alpha_3 x_3)+a_{12}x_2)+b_{22}v)$$

On the other hand, the vector v does not belong in $L(x_1,x_2,x_3,x_4)$, so according to that $v\neq\beta_1 x_1+\beta_2 x_2+\beta_3 x_3+\beta_4 x_4$, for any $\beta_1,\beta_2,\beta_3,\beta_4$ from the field Φ. According to this, for the vector v we have a form $v=x+\alpha y$, where $x=\beta_1 x_1+\beta_2 x_2+\beta_3 x_3+\beta_4 x_4$ for fixed scalars $\beta_1,\beta_2,\beta_3,\beta_4$, and y is a vector which is not from $L(x_1,x_2,x_3,x_4)$.

Now, it is unclear if the vector u is a loop element.

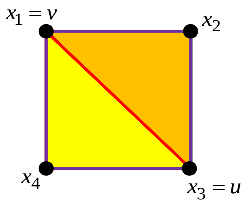
From the definition of this 2-subspace is clear that (u,v) is its element. But, also an element is the 2-vector (u,x_2), too, which can be presented in the form

$$(u,x_2)=A(\alpha_1 x_1+\alpha_3 x_3,x_2)^T=\begin{bmatrix}a_{11} & a_{12}\\ 0 & 1\end{bmatrix}(\alpha_1 x_1+\alpha_3 x_3,x_2)^T=(\underbrace{a_{11}(\alpha_1 x_1+\alpha_3 x_3)+a_{12}x_2}_{u},x_2).$$

This 2-vector together with the 2-vector (x_2,x_1) form a finite branch 2-subspace which we had at the beginning of the discussion. Here finally we have that the vector u is a loop of two elements from M', or it can be considered as a finite branch 2-subspace.

Comment. Sub case 3 is of course much more complicated than sub case 2.

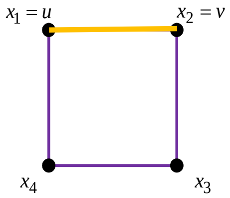
Sub cases of case 3



Sub case 1. $u=x_i$, $v=x_{i+1}$ for some $i\in\{1,2,..n\}$.

In this case we do not have extension. According to the definition of M and the definition of the 2-subspace generated by the element $(u,v)\equiv(x_i,x_{i+1})$, i.e. $M=M'$

Sub case 2. $u=x_i$, $v=x_{i+2}$ where without loss of generality we can assume that $i+2\leq n$.



Especially, only in this case, with addition of one element we get two 2-subspaces which are kernel 2-subspaces and which touch each other. In this case, as shown in the drawing, only one element is enough (in this case it is (x_2,x_4) so that we get a complete kernel 2-subspace). Now it is important to determine the general type of elements from this new 2-subspace.

We will consider now the two kernel 2-subspaces, generated by $(u,x_2),(x_2,v),(v,u)$ and $(u,x_4),(x_4,v),(v,u)$ separately, as well as the loop 2-subspaces which are 2-subspaces from the new 2-subspace M', generated by $(x_3,x_2),(x_3,x_4),(x_3,x_1)$ and $(x_1,x_2),(x_1,x_4),(x_1,x_3)$.

Situation a)

A kernel 2-subspace generated by $(u,x_2),(x_2,v),(v,u)$, i.e. by $(x_3,x_2),(x_2,x_1),(x_1,x_3)$.

The elements from the kernel 2-subspace generated by $(u,x_2),(x_2,v),(v,u)$, because of the way of denotation and formation have the form $(\alpha_1 x_1+\alpha_2 x_2+\alpha_3 x_3,\beta_1 x_1+\beta_2 x_2+\beta_3 x_3)$.

Situation b)

A kernel 2-subspace generated by $(u,x_4),(x_4,v),(v,u)$, i.e. by $(x_3,x_4),(x_4,x_1),(x_1,x_3)$

Again because of the way of denotation and formation of the elements of this 2-subspace, we have that they have the following form $(\gamma_1 x_1+\gamma_3 x_3+\gamma_4 x_4, \delta_1 x_1+\delta_3 x_3+\delta_4 x_4)$.

Between this two elements we can determine addition only in the case when

i) $\alpha_2=0$ and $\gamma_4=0$.

ii) $\beta_2=0$ and $\delta_4=0$.

Due to symmetry it is enough to consider only one of this cases.

Sub case *i*)

Let $\alpha_2=0$ and $\alpha_4=0$. Then, we have that the elements get the form

$$(\alpha_1 x_1+\alpha_3 x_3, \beta_1 x_1+\beta_2 x_2+\beta_3 x_3)$$

$$(\alpha_1 x_1+\alpha_3 x_3, \beta_1 x_1+\beta_3 x_3+\beta_4 x_4)$$

If we apply addition between them, we get that the sum is equal to

$$(\alpha_1 x_1+\alpha_3 x_3, \beta_1 x_1+\beta_2 x_2+\beta_3 x_3)+(\alpha_1 x_1+\alpha_3 x_3, \delta_1 x_1+\delta_3 x_3+\delta_4 x_4)=$$
$$=(\alpha_1 x_1+\alpha_3 x_3, (\beta_1+\delta_1)x_1+\beta_2 x_2+(\beta_3+\delta_3)x_3+\delta_4 x_4)$$

or has a form

$$A(\alpha_1 x_1+\alpha_3 x_3, (\beta_1+\delta_1)x_1+\beta_2 x_2+(\beta_3+\delta_3)x_3+\delta_4 x_4),$$

for arbitrary matrix $A\in M_2(\Phi)$.

Sub case *ii)*

It is completely analogous to the sub case *i*).

Now, we will consider the loop 2-subspaces, in which as loops appear the elements x_1 and x_3. Certainly, between these two 2-subspaces it can happen that addition may be done, because both of them as its own 2-subspace have the 2-subspace $M=\{(A(x_1,x_3)^T / A\in M_2(\Phi)\}$. But now, in this situation we have the following possibilities:

- if we have two elements and both of them belong in the loop with a loop centre x_1, where they can be added, then, their sum is again an element of that loop. And the sum is either in the first kernel 2-subspace, or in the other kernel 2-subspace.

- If one of them belongs in the loop in x_1 and the other one in the loop of x_1 then the sum belongs in the 2-subspace generated from (x_1,x_3), i.e. in $\{A(x_1,x_3)/A\in M_2(\Phi)\}$

In some way, it seems as if we have a basis in the 2-dimensional subspace (like a line in a bundle of planes) and on it, for every point on it as if four dimensional subspaces are set. From every one of them, only two points are taken and are built four dimensional subspaces. All together make this 2-subspace.

For example, the element (x_2,x_4) does not belong in this new 2-subspace M'. But then, in this 2-dimensional subspace does not belong not anyone of the elements in the following form

$$A(x_2,x_4)=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}(x_2,x_4)=(a_{11}x_2+a_{12}x_4,a_{21}x_2+a_{22}x_4),$$

for any $A\in M_2(\Phi)$.

Sub case 3. $u=x_i$, $v=x_{i+k}$ for some $k>2$, where without loss of generality we can assume that $i+k\le n$. If $i+k>n$, then we consider it that $i+k=p$, $p<i-1$ where $i+k\equiv p \pmod n$.

This case for $n=4$ is not possible at all, because for $k=3$ and $i=1$ we get the vectors x_1 and x_4 and it comes to the case 3.1. (sub case 3.1.) . All other similar situations are considered analogously.

Sub case 4. $u\in L(\alpha_1x_1+\alpha_3x_3,x_2)\times L(\alpha_1x_1+\alpha_3x_3,x_2)$, $v=x_4$.

Analogously to this case are considered also the following cases:

a) $u\in L(\alpha_2x_2+\alpha_4x_4,x_3)\times L(\alpha_2x_2+\alpha_4x_4,x_3)$, $v=x_1$

b) $u\in L(\alpha_1x_1+\alpha_3x_3,x_4)\times L(\alpha_1x_1+\alpha_3x_3,x_4)$, $v=x_2$

c) $u\in L(\alpha_2x_2+\alpha_4x_4,x_1)\times L(\alpha_2x_2+\alpha_4x_4,x_1)$, $v=x_3$

In this situation, it is clear to consider the case that is given initially. Here we should note that the vector u has the form of a real coordinate from the 2-vector

$$A(\alpha_1x_1+\alpha_3x_3,x_2)^T=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}(\alpha_1x_1+\alpha_3x_3,x_2)=(\underbrace{a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2}_{u},\underbrace{a_{21}(\alpha_1x_1+\alpha_3x_3)+a_{22}x_2}_{w}).$$

Now, it is obvious that the vector $u=a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2$ and the vector $v=x_4$ are linearly independent. So, the pair (u,v) until now did not a part of M .

It is worth mentioning that the 2-vector (u,x_2) is a 2-vector which belongs in the 2-subspace M . Indeed, from the assumptions that $\begin{cases}(x_1,x_2)\\(x_3,x_2)\end{cases}\in M$, we get that $\begin{cases}\alpha_1(x_1,x_2)=(\alpha_1x_1,x_2)\\ \alpha_3(x_3,x_2)=(\alpha_3x_3,x_2)\end{cases}\in M$, and because M is a 2-subspace, we get that $(u,x_2)=(\alpha_1x_1+\alpha_3x_3,x_2)\in M$.

Now, it is clear that $(u,x_2),(x_2,x_3),(x_3,x_4=v)\in M$, so, adding of (u,v) we get a cycle from four 2-vectors. They form for themselves a cycle 2-subspace. Parallel to this cycle 2-subspace we have one more cycle 2-subspace, generated by the 2-vectors $(u,x_2),(x_2,x_1),(x_1,v=x_4),(v=x_4,u)$. If the new cycle 2-subspaces are denoted with S and S', we will have that

$M'=M\cup S\cup S'$

Sub case 5. $u\in L(\alpha_1x_1+\alpha_3x_3,x_2)\times L(\alpha_1x_1+\alpha_3x_3,x_2)$, $v=\alpha_3x_3+\alpha_4x_4$.

All other analogous cases, which can be gotten with cyclic shift of the indexes are considered analogously to this case. Those cases are:

a) $u\in L(\alpha_2x_2+\alpha_4x_4,x_3)\times L(\alpha_2x_2+\alpha_4x_4,x_3)$, $v=\beta_1x_1+\beta_4x_4$

b) $u\in L(\alpha_3x_3+\alpha_1x_1,x_4)\times L(\alpha_3x_3+\alpha_1x_1,x_4)$, $v=\beta_1x_1+\beta_2x_2$

c) $u\in L(\alpha_2x_2+\alpha_4x_4,x_1)\times L(\alpha_2x_2+\alpha_4x_4,x_1)$, $v=\beta_2x_2+\beta_3x_3$

In this situation it is important to consider the initial case, where we will get the vector u from the equality

$$A(\alpha_1x_1+\alpha_3x_3,x_2)^T=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}(\alpha_1x_1+\alpha_3x_3,x_2)=(\underbrace{a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2}_{u},\underbrace{a_{21}(\alpha_1x_1+\alpha_3x_3)+a_{22}x_2}_{w}),$$

where from we see that $u=a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2$ is a coordinate of a 2-vector from M'. Additionally, $(\underbrace{a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2}_{u},x_2)$ we have it in M. It is clear that this vector without adding of $(u,v=x_3)$, as we said, we have it in M, because

$$A(\alpha_1x_1+\alpha_3x_3,x_2)^T=\begin{bmatrix}a_{11} & a_{12}\\ 0 & 1\end{bmatrix}(\alpha_1x_1+\alpha_3x_3,x_2)=(\underbrace{a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2}_{u},x_2).$$

Now $v=\alpha_3x_3+\alpha_4x_4$ so, according to this the pair (u,v) is consisted of linearly independent elements. It remains to see what kind of structure will build all this vectors.

But, now, it is important to see what will happen with the vectors $u=a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2,x_2,x_3,v=\alpha_3x_3+\alpha_4x_4$. This four vectors are linearly independent. Indeed they form a cyclic 2-subspace.

So, according to that, the 2-vectors $(u,x_2),(x_2,x_3),(x_3,v),(v,u)$ are vectors which all of them belong in the new 2-subspace M', so, they for themselves form a cyclic 2-subspace built with four elements, and we will denote it with S.

Similarly, it is worth considering also the vectors $u=a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2,x_2,x_1,x_4,v=\alpha_3x_3+\alpha_4x_4$ (which are not linearly independent), and of course the 2-vectors $(u,x_2),(x_2,x_1),(x_1,x_4),(x_4,v),(v,u)$, which are five and form cyclic 2-subspace M' with five elements which we will denote with S'. Now it is clear that

$M'=M\cup S\cup S'$. .

Sub case 6. $u\in L(\alpha_1x_1+\alpha_3x_3,x_2)\times L(\alpha_1x_1+\alpha_3x_3,x_2)$, $v=x_3$.

Analogously to this case, is considered also the case when the vector $u\in L(\alpha_1x_1+\alpha_3x_3,x_2)\times L(\alpha_1x_1+\alpha_3x_3,x_2)$ remains the same and $v=x_1$. Analogously to this case are considered also the following cases:

a) $u\in L(\alpha_2x_2+\alpha_4x_4,x_3)\times L(\alpha_2x_2+\alpha_4x_4,x_3)$, $v=x_2$, i.e.
$u\in L(\alpha_2x_2+\alpha_4x_4,x_3)\times L(\alpha_2x_2+\alpha_4x_4,x_3)$, $v=x_4$

b) $u\in L(\alpha_1x_1+\alpha_3x_3,x_4)\times L(\alpha_1x_1+\alpha_3x_3,x_4)$, $v=x_1$, i.e.
$u\in L(\alpha_1x_1+\alpha_3x_3,x_4)\times L(\alpha_1x_1+\alpha_3x_3,x_4)$, $v=x_3$

c) $u\in L(\alpha_2x_2+\alpha_4x_4,x_1)\times L(\alpha_2x_2+\alpha_4x_4,x_1)$, $v=x_2$,
$u\in L(\alpha_2x_2+\alpha_4x_4,x_1)\times L(\alpha_2x_2+\alpha_4x_4,x_1)$, $v=x_4$.

Now, it is clear that it is enough to consider only the initial case from this vectors, mentioned in the beginning. Certainly, in this case most valuable is to consider the case when $\alpha_1 \neq 0, \alpha_3 \neq 0$. We can get the vector u in this case as a member of the 2-vector from the equality

$$A(\alpha_1 x_1 + \alpha_3 x_3, x_2)^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} (\alpha_1 x_1 + \alpha_3 x_3, x_2) = (\underbrace{a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12} x_2}_{u}, \underbrace{a_{21}(\alpha_1 x_1 + \alpha_3 x_3) + a_{22} x_2}_{w}),$$

i.e. $u = a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12} x_2$, where $a_{11}, \alpha_1, \alpha_3, a_{12}$ are arbitrary, but fixed scalars from Φ .

Now, u, x_2, x_3 are three vectors which already belong in M , as coordinates of 2-vectors. But, we have a situation here when $(u, v = x_3)$ as 2-vector, we insert the 2-vector (x_3, x_2) which already exists in M , and it is a question whether the 2-vector $(\underbrace{a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12} x_2}_{u}, x_2)$ we have in M . It is clear that also that vector without adding of $(u, v = x_3)$, we already have it in M , because

$$A(\alpha_1 x_1 + \alpha_3 x_3, x_2)^T = \begin{bmatrix} a_{11} & a_{12} \\ 0 & 1 \end{bmatrix} (\alpha_1 x_1 + \alpha_3 x_3, x_2) = (\underbrace{a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12} x_2}_{u}, x_2).$$

According to this, the 2-vectors $(u, v = x_3)$, (x_3, x_2) and $(\underbrace{a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12} x_2}_{u}, x_2)$ build a kernel 2-subspace in M', which we will denote with S .

According to this, now

$M' = M \cup L(x_2, x_3, u) \times L(x_2, x_3, u)$

Now, it is worth to consider also the special cases of this case, i.e. "playing" with α_1 and α_3 in already made signature, in order to get more precise picture in this case.

Situation 1. $\alpha_1 = 0, \alpha_3 \neq 0$.

In this situation we have that the vector $u = a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12} x_2$ gets the form $u = a_{11}\alpha_3 x_3 + a_{12} x_2$, and now in fact there is no extension of M , and the vectors $v = x_3, u = a_{11}\alpha_3 x_3 + a_{12} x_2, x_2$ are three linearly independent vectors.

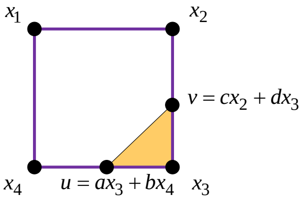
Situation 2. $\alpha_1 =\neq 0, \alpha_3 = 0$.

In this situation, we have a vector which has the form $u = a_{11}\alpha_1 x_1 + a_{12} x_2$. Now, we have the same situation as in the sub case 9. So, we have connected two kernel 2-subspaces from already existing elements from M , i.e.

$M' = L(x_1, x_3, x_2) \times L(x_1, x_3, x_2) \cup L(x_3, x_4, x_1) \times L(x_3, x_4, x_1)$

Sub case 7. $u = \alpha_3 x_3 + \alpha_4 x_4$, $v = \alpha_2 x_2 + \alpha_3 x_3$,

In this case it is clear that the vectors $u = \alpha_3 x_3 + \alpha_4 x_4$, x_3 and $v = \alpha_2 x_2 + \alpha_3 x_3$ are linearly independent vectors, and the pairs $(u, x_3), (x_3, v), (v, u)$ belong in the new 2-subspace. But, these 2-vectors for themselves form a kernel 2-subspace. The form of this kernel 2-subspace is $\{(\gamma_1 u + \delta_1 v + \lambda_1 x_3, \gamma_2 u + \delta_2 v + \lambda_2 x_3) / \gamma_i, \delta_i, \lambda_i \in \Phi\}$.



But, let us note here that (u,v) and (u,x_3) are two 2-vectors that belong in M', so, according to that

$$\begin{bmatrix}1 & 0\\ 0 & \frac{1}{\alpha_2}\end{bmatrix}\left((u,v)+\begin{bmatrix}1 & 0\\ 0 & -\alpha_3\end{bmatrix}(u,x_3)\right)=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{\alpha_2}\end{bmatrix}\big((u,\alpha_2x_2+\alpha_3x_3)+(u,-\alpha_3x_3)\big)=$$

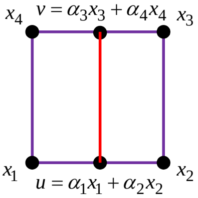
$$=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{\alpha_2}\end{bmatrix}(u,\alpha_2x_2)=(u,x_2)\quad ,$$

because (v,u) and (x_3,u) belong in M' that also the 2-vector (x_2,u) belongs in M' Now, according the sub case 3.9 we get that also the 2-vector (x_2,x_4) also belongs in M'. Now, it is clear that this sub case comes to the sub case 3.2, i.e. in this situation the 2-subspaces which are kernel 2-subspaces and one of them is generated by $(x_1,x_2),(x_2,x_4),(x_4,x_1)$, and the other is generated by $(x_2,x_3),(x_3,x_4),(x_4,x_2)$ are glued one to another. Their common 2-subspace is the 2-subspace generated with the 2-vector (x_2,x_4).

Sub case 8. $u=\alpha_1x_1+\alpha_2x_2$, $v=\beta_3x_3+\beta_4x_4$, where $\alpha_1\alpha_2\neq 0$, $\beta_3\beta_4\neq 0$

Completely analogous to this case is

a) $u=\alpha_2x_2+\alpha_3x_3$, $v=\beta_1x_1+\beta_4x_4$



Now, we have a situation in which the vectors x_1,u,v,x_4 are linearly independent. They, for themselves form a cyclic 2-subspace. Completely analogous, in the same time also the vectors u,x_2,x_3,v are linearly independent. So, according to that, they also for themselves form a cyclic 2-subspace. The basic question is what kind of relationship have these cyclic 2-subspaces. Certainly u and v are loops of the loop 2-subspaces. So,

$$M'=M\cup S\cup S'\cup K\cup K',$$

where S,S' are loop/ sub loop 2-subspaces, and K,K' are the cyclic 2-subspaces.

From technical point of view, the 2-vectors $(v,x_4),(v,x_3),(v,u)$ are three 2-vectors from the 2-vector subspace M'. So, they build one loop 2-subspace with loop centre v (completely analogous is for the vector u). Only disputable is the fact that x_3,v,x_4 are linearly dependent vectors.

On the other hand, for any δ and γ we would have that the 2-vector

$$(\delta u,\gamma v)=\begin{bmatrix}\delta & 0\\ 0 & \gamma\end{bmatrix}(u,v)\in M' \ .$$

But, since $(u,x_2),(u,x_1),(v,x_3),(v,x_4)\in M$ and with that also in M', we have that

$$(\delta v,x_4)=\begin{bmatrix}\delta & 0\\ 0 & 1\end{bmatrix}(v,x_4)\,,\ (\delta v,x_3)=\begin{bmatrix}\delta & 0\\ 0 & 1\end{bmatrix}(v,x_3)\,,$$

$$(\gamma u, x_1)=\begin{bmatrix}\gamma & 0\\ 0 & 1\end{bmatrix}(u,x_1)\,,\ (\gamma u, x_2)=\begin{bmatrix}\gamma & 0\\ 0 & 1\end{bmatrix}(u,x_2)$$

are all 2-vectors from M'. But, then we have that

$(\gamma u, x_2),(x_2,x_3),(x_3,\delta v),(\delta v,\gamma u)$ and $(\gamma u,x_1),(x_1,x_4),(x_4,\delta v),(\delta v,\gamma u)$,

are two groups both of four 2-vectors, and every group makes one cycle. Practically, with the vectors u and v, i.e. with their one dimensional subspaces generated in the corresponding subspaces from X in one dimension, are determined sequence of cyclic 2-subspaces which are glued one to another. In the place where they are glued, i.e. in their endings δu and γv are determined one loop 2-subspace each.

Sub case 9. $u=\alpha_1 x_1+\alpha_2 x_2$, $v=x_3$.

In this situation, we have three vectors $u=\alpha_1 x_1+\alpha_2 x_2, x_2, v=x_3$ which are linearly independent. Because the 2-vectors $(u,x_2),(x_2,v=x_3),(v,u)$ belong in the new 2-subspace, we get that the kernel 2-subspace generated from these three 2-vectors is a 2-subspace consisted in it.

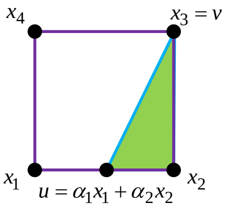
But, let us note here that $(\alpha_1 x_1+\alpha_2 x_2, x_3),(x_2,x_3)\in M'$, and M' is a 2-subspace, we get that

$$\begin{bmatrix}\dfrac{1}{\alpha_1} & 0\\ 0 & 1\end{bmatrix}\left[(\alpha_1 x_1+\alpha_2 x_2,x_3)+\begin{bmatrix}-\alpha_2 & 0\\ 0 & 1\end{bmatrix}(x_2,x_3)\right]=\begin{bmatrix}\dfrac{1}{\alpha_1} & 0\\ 0 & 1\end{bmatrix}\left[(\alpha_1 x_1+\alpha_2 x_2,x_3)+(-\alpha_2 x_2,x_3)\right]=$$

$$=\begin{bmatrix}\dfrac{1}{\alpha_1} & 0\\ 0 & 1\end{bmatrix}(\alpha_1 x_1+\alpha_2 x_2-\alpha_2 x_2,x_3)=\begin{bmatrix}\dfrac{1}{\alpha_1} & 0\\ 0 & 1\end{bmatrix}(\alpha_1 x_1,x_3)=(x_1,x_3)$$

i.e. $(x_1,x_3)\in M'$.

Now, we have a situation $(x_1,x_3),(x_3,x_2),(x_2,x_1)\in M'$, i.e. here a whole one kernel 2-subspace is consisted in M'. Similarly, $(x_1,x_3),(x_3,x_4),(x_4,x_1)\in M'$, so, according to this, we have a situation completely the same as in the sub case 3.2.



Sub case 10.

$u\in L(\alpha_1 x_1+\alpha_3 x_3,x_2)\times L(\alpha_1 x_1+\alpha_3 x_3,x_2)$, $v\in L(\alpha_2 x_2+\alpha_4 x_4,x_3)\times L(\alpha_2 x_2+\alpha_4 x_4,x_3)$.

We have three more completely analogous 2-subspaces of the 2-subspace from this case, i.e.

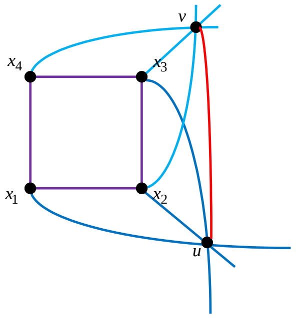
a) $u\in L(\alpha_2 x_2+\alpha_4 x_4,x_3)\times L(\alpha_2 x_2+\alpha_4 x_4,x_3)$, $v\in L(\alpha_3 x_3+\alpha_1 x_1,x_4)\times L(\alpha_3 x_3+\alpha_1 x_1,x_4)$

b) $u\in L(\alpha_1 x_1+\alpha_3 x_3,x_4)\times L(\alpha_1 x_1+\alpha_3 x_3,x_4)$, $v\in L(\alpha_2 x_2+\alpha_4 x_4,x_1)\times L(\alpha_2 x_2+\alpha_4 x_4,x_1)$

c) $u\in L(\alpha_2 x_2+\alpha_4 x_4,x_1)\times L(\alpha_2 x_2+\alpha_4 x_4,x_1)$, $v\in L(\alpha_1 x_1+\alpha_3 x_3,x_2)\times L(\alpha_1 x_1+\alpha_3 x_3,x_2)$.

That is why, it is enough to consider only the first case.

It is clear that the 2-vectors (x_3,v) and (u,x_2) belong in the new 2-subspace M'. But then we have that (v,x_3), (x_2,x_3), (x_2,u) and (u,v) are four vectors from the new subspace, so, according to the definition of cyclic 2-subspace, they form such cyclic 2-subspace. But then we would have in fact two cyclic 2-subspaces, and one of them is the starting 2-subspace. Their common part is the 2-vector (x_2,x_3).



Practically, we have here two cyclic 2-subspaces, which have one common 2-subspace $\{A(x_2,x_3)/A\in M_2(\Phi)\}$. Let us denote this subspace with S. According to this, we have an extension from the type

$M'=M\cup S$.

The previous considerations are in case when $\alpha_1,\alpha_2,\alpha_3,\alpha_4\neq 0$.

Now, we will consider situations when some of this scalars may be zero. Then, we will have some more specific cases. In fact we will deal with scalars which are zero, and belong in one of these two subspaces. We will choose the first of them.

Situation 1. $\alpha_1=0$

In general case $u=a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2$, but in this situation it is in the form $u=a_{11}\alpha_3x_3+a_{12}x_2$, i.e. without loss of generality we can consider that it is in the form $u=\alpha_2x_2+\alpha_3x_3$. Now, it is clear that in this situation, we can get the 2-vector (v,x_2) as a 2-combination of the 2-vectors (v,x_3) and (v,u), in the following way:

$$\begin{bmatrix}1&0\\0&\frac{1}{\alpha_2}\end{bmatrix}\left(\begin{bmatrix}1&0\\0&-\alpha_3\end{bmatrix}(v,x_3)+(v,\alpha_2x_2+\alpha_3x_3)\right)=\begin{bmatrix}1&0\\0&\frac{1}{\alpha_2}\end{bmatrix}\left((v,-\alpha_3x_3)+(v,\alpha_2x_2+\alpha_3x_3)\right)=$$

$$=\begin{bmatrix}1&0\\0&\frac{1}{\alpha_2}\end{bmatrix}(v,\alpha_2x_2+\alpha_3x_3-\alpha_3x_3)=\begin{bmatrix}1&0\\0&\frac{1}{\alpha_2}\end{bmatrix}(v,\alpha_2x_2)=(v,x_2)$$

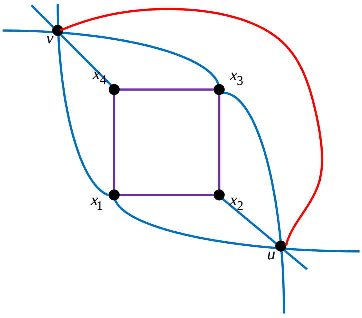
So, the 2-vectors $(v,x_3),(x_3,x_2),(x_2,v)$ are three vectors which belong in M', so, according to that, the kernel 2-subspace generated by them also belongs in M'. In this situation, we have connected one cycle 2_subspace (which in fact is the starting one) and one kernel 2-subspace (which we described totally) i.e..

$M'=M\cup S$.

Situation 2. $\alpha_3=0$

In general case $u=a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2$, but in this situation it is in the form $u=a_{11}\alpha_1x_1+a_{12}x_2$, i.e. without loss of generality, we can consider that it has the form $u=\alpha_2x_2+\alpha_1x_1$. In this situation we have that the four vectors $(u,x_2),(x_2,x_3),(x_3,v),(v,u)$ form a cyclic 2-subspace, which is a part from M'. We will denote it with S. So,

$M' = M \cup S$.



Sub case 11.

$u \in L(\alpha_1 x_1 + \alpha_3 x_3, x_2) \times L(\alpha_1 x_1 + \alpha_3 x_3, x_2)$,

$v \in L(\alpha_1 x_1 + \alpha_3 x_3, x_4) \times L(\alpha_1 x_1 + \alpha_3 x_3, x_4)$.

Completely analogous to this 2-subspace there is one more 2-subspace and it is determined in the following way:

$u \in L(\alpha_2 x_2 + \alpha_4 x_4, x_3) \times L(\alpha_2 x_2 + \alpha_4 x_4, x_3)$,

$v \in L(\alpha_2 x_2 + \alpha_4 x_4, x_1) \times L(\alpha_2 x_2 + \alpha_4 x_4, x_1)$

In this sub case we have two cyclic 2-subspaces which are generated each with five elements, i.e. those 2-subspaces are generated by

a) $(v,x_4),(x_4,x_1),(x_1,x_2),(x_2,u),(u,v)$, denoted as S

b) $(v,x_4),(x_4,x_3),(x_3,x_2),(x_2,u),(u,v)$, denoted as S'.

Now, it is clear that

$M' = M \cup S \cup S'$.

Sub case 12. In this sub case we have a situation when both vectors u and v belong in $L(x_1,x_2,x_3,x_4)$, but they are not a part from neither 2-subspace from the form

$L(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}, x_i) \times L(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}, x_i)$,

for $i = 1,2,3,4$. This situation even though is possible, technically is feasible, but still the result is as in the case 1.

Sub case 13. $u \in L(\alpha_1 x_1 + \alpha_3 x_3, x_2) \times L(\alpha_1 x_1 + \alpha_3 x_3, x_2)$, $v = \alpha_2 x_2 + \alpha_3 x_3$

In this sub case we have the 2-vectors $(u,x_2),(x_2,v),(v,u)$, which form for themselves a kernel 2-subspace from M', which will be denoted as S. So, according to this we have that

$M' = M \cup S$.

Now, we will consider this kernel 2-subspace. Let us note that $(u,x_2),(u,v) = (u,\alpha_2 x_2 + \alpha_3 x_3)$ are two vectors from M'. But, then we have that

$$\begin{bmatrix}1 & 0\\ 0 & \dfrac{1}{\alpha_3}\end{bmatrix}\left((u,\alpha_2 x_2 + \alpha_3 x_3) + \begin{bmatrix}1 & 0\\ 0 & -\alpha_2\end{bmatrix}(u,x_2)\right) = \begin{bmatrix}1 & 0\\ 0 & \dfrac{1}{\alpha_3}\end{bmatrix}\big((u,\alpha_2 x_2 + \alpha_3 x_3) + (u,-\alpha_2 x_2)\big) =$$

$$= \begin{bmatrix}1 & 0\\ 0 & \dfrac{1}{\alpha_3}\end{bmatrix}(u,\alpha_2 x_2 + \alpha_3 x_3 - \alpha_2 x_2) = \begin{bmatrix}1 & 0\\ 0 & \dfrac{1}{\alpha_3}\end{bmatrix}(u,\alpha_3 x_3) = (u,x_3) \in M'$$

According to this, the loop 2-subspace is generated from the 2-vectors $(u,x_2),(x_2,x_3),(x_3,u)$.

In the following part we will present one elaborated case of extension of 2-subspace and extension of a 2-skew-symmetric form defined on it.

3. EXTENSION OF A 2-SKEW-SYMMETRIC LINEAR FORM

Theorem. *Let* $\Lambda : M \to \mathbb{R}$ *be a* 2*-skew-symmetric form such that* $\Lambda(x,y) \le p(x,y)$ *for every* $(x,y) \in M$ *,* $p : X^2 \to \mathbb{R}$ *be a* 2*-semi norm and* M *is a branch* 2*-subspace of the* 2*-space* X^2*. Let* M' *be an extension of* M *as in sub case 1 of case 2. Then there exists a* 2*-skew-symmetric linear form* $\Lambda' : M' \to \mathbb{R}$ *such that*

$\Lambda' / M = \Lambda$

$-p(-x,y) \le \Lambda(x,y) \le p(x,y)$. (*)

Proof. In this theorem, as before we will choose two arbitrary elements from the 2-subspace M , which at the same time belong also in the loop u . Let that be the elements $(\alpha_2 x_2 + \alpha_4 x_4, u)$ and $(\beta_2 x_2 + \beta_4 x_4, u)$. For 2-skew-symmetric form Λ , according to the conditions of the theorem we have that

$$\Lambda(\alpha_2 x_2 + \alpha_4 x_4, u) + \Lambda(\beta_2 x_2 + \beta_4 x_4, u) = \Lambda(\alpha_2 x_2 + \alpha_4 x_4 + \beta_2 x_2 + \beta_4 x_4, u) \le$$
$$\le p(\alpha_2 x_2 + \alpha_4 x_4 + \beta_2 x_2 + \beta_4 x_4, u) = p(\alpha_2 x_2 + \alpha_4 x_4 - v + \beta_2 x_2 + \beta_4 x_4 + v, u) \le$$
$$\le p(\alpha_2 x_2 + \alpha_4 x_4 - v, u) + p(\beta_2 x_2 + \beta_4 x_4 + v, u)$$

In other words, the inequality holds

$$\Lambda(\alpha_2 x_2 + \alpha_4 x_4, u) - p(\alpha_2 x_2 + \alpha_4 x_4 - v, u) \le p(\beta_2 x_2 + \beta_4 x_4 + v, u) - \Lambda(\beta_2 x_2 + \beta_4 x_4, u) .$$

Since $\alpha_2, \alpha_4 \in \mathbb{R}$ and $\beta_2, \beta_4 \in \mathbb{R}$ are arbitrary, we get that

$$\sup_{\alpha_2,\alpha_4} \Lambda(\alpha_2 x_2 + \alpha_4 x_4, u) - p(\alpha_2 x_2 + \alpha_4 x_4 - v, u) = d \le p(\beta_2 x_2 + \beta_4 x_4 + v, u) - \Lambda(\beta_2 x_2 + \beta_4 x_4, u)$$

According to that, for arbitrary $\alpha_2, \alpha_4, \beta_2, \beta_4 \in \mathbb{R}$, the inequalities hold

$$\Lambda(\alpha_2 x_2 + \alpha_4 x_4, u) - p(\alpha_2 x_2 + \alpha_4 x_4 - v, u) \le d$$
$$d \le p(\beta_2 x_2 + \beta_4 x_4 + v, u) - \Lambda(\beta_2 x_2 + \beta_4 x_4, u)$$

i.e.

$$\Lambda(\alpha_2 x_2 + \alpha_4 x_4, u) - d \le p(\alpha_2 x_2 + \alpha_4 x_4 - v, u) \qquad (1)$$
$$\Lambda(\beta_2 x_2 + \beta_4 x_4, u) + d \le p(\beta_2 x_2 + \beta_4 x_4 + v, u) \qquad (2)$$

Now, we will determine $\Lambda' : M' \to \mathbb{R}$ with

$$\Lambda'[A(\alpha_2 x_2 + \alpha_4 x_4 + \gamma v, u)] = (\det A)[\Lambda(\alpha_2 x_2 + \alpha_4 x_4, u) + \gamma d] ,\ \gamma \in \mathbb{R} ,$$
$$\Lambda'(x,y)] = \Lambda(x,y) ,\ (x,y) \in M .$$

According to this $\Lambda' / M = \Lambda$.

On the other hand, if in (1) instead α_2 and α_4 we choose $\frac{\alpha_2}{t}$ and $\frac{\alpha_4}{t}$, $t > 0$ and if we use the properties of Λ and p respectively, we get that

$$\Lambda(\alpha_2 x_2 + \alpha_4 x_4, u) - td \le p(\alpha_2 x_2 + \alpha_4 x_4 - tv, u) . \qquad (3)$$

Completely analogous, if in (2) instead β_2 and β_4 we choose $\frac{\beta_2}{t}$ and $\frac{\beta_4}{t}$, $t > 0$ respectively, and again if we use the properties of Λ and p, we get that

$$\Lambda(\alpha'_{i-1} x_{i-1} + \alpha'_{i+1} x_{i+1}, u) + td \le p(\alpha_{i-1} x_{i-1} + \alpha_{i+1} x_{i+1} + tv, u) . \qquad (4)$$

Now, from (3) and (4) we see that

$$\Lambda'(\beta_2 x_2 + \beta_4 x_4 + \gamma v, u) \le p(\beta_2 x_2 + \beta_4 x_4 + \gamma v, u),$$

where from it is clear that in general case $\Lambda' \le p$ on M'. In other words the inequality (*) holds.

CONFLICT OF INTEREST

No conflict of interest was declared from the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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