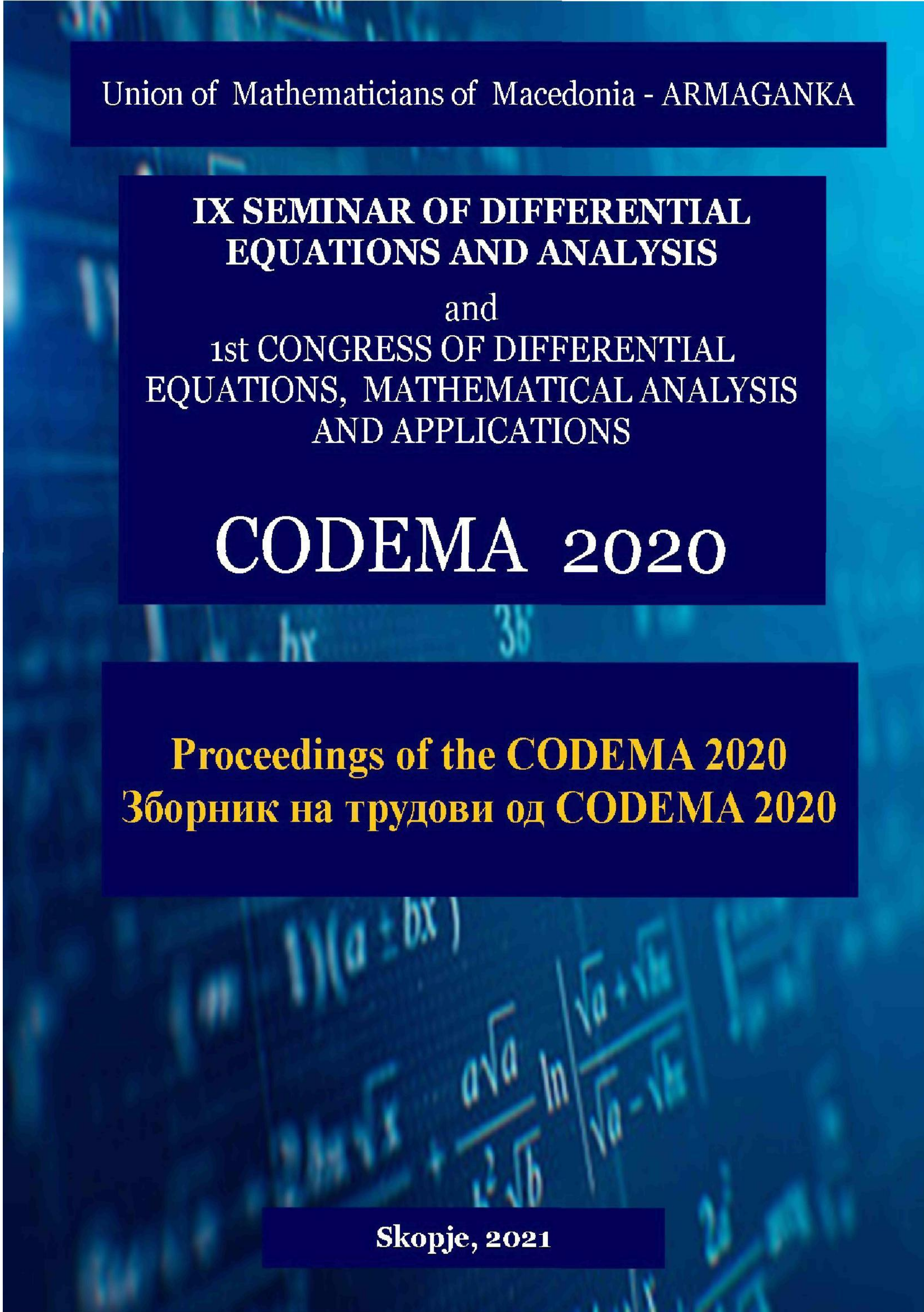


Union of Mathematicians of Macedonia - ARMAGANKA

IX SEMINAR OF DIFFERENTIAL EQUATIONS AND ANALYSIS

and

1st CONGRESS OF DIFFERENTIAL EQUATIONS, MATHEMATICAL ANALYSIS AND APPLICATIONS

CODEMA 2020

Proceedings of the CODEMA 2020
Зборник на трудови од CODEMA 2020

Skopje, 2021

EXTENSION OF FINITE BRANCH 2-SUBSPACE AND SOME EXTENSIONS OF HAHN - BANACH TYPE FOR SKEW-SYMMETRIC 2-LINEAR FUNCTIONALS DEFINED ON IT

ISBN 978-608-4904-09-0 **UDC:** 517.982.22:515.173

Slagjana Brsakoska[1], Aleksa Malcheski[2]

Abstract. In this paper 2-subspaces from 2-space X^2, which are from finite branch 2-subspace type, generated with four 2-vectors, will be taken in consideration. Then all its possible extensions adding one element (u,v) and their complete description will be considered. Also, all extensions of 2-skew-symmetric linear form defined on 2-subspace M' of Hahn-Banach type will be considered, in the cases when one vector belongs in 2-vector from M, and the other does not belong in any 2-vector from M, as well as cases when the two coordinates of (u,v) do not belong in M.

1. INTRODUCTION

Extensions of mappings is something that is often looked at in various mathematical disciplines. One classical example of extension of a given mapping is of course the Hanh-Banach theorem for linear functionals. One version of it comprises the contents of the following theorem.

Theorem 1. *Let* M *be a vector subspace of the vector space* X. *The functional* $p: X \to \mathbb{R}$ *satisfies the conditions*

(a) $p(x+y) \leq p(x) + p(y)$

(b) $p(tx) = tp(x)$,

for every $x, y \in X$ *and* $t \geq 0$.

The functional $f: M \to R$ *is linear and* $f(x) \leq p(x)$. *There exists a linear functional* $\Lambda: X \to \mathbb{R}$ *such that* $\Lambda / M = f$ *and* $-p(-x) \leq \Lambda(x) \leq p(x)$.

Of course, it is worth mentioning here both the definitions for 2-norm and especially for 2 semi norm, which we will use many times further.

Definition 1. Let X be a vector space over the field Φ. The mapping $\| \bullet, \bullet \|: X^2 \to \mathbb{R}_{\geq 0}$ for which the following conditions are fulfilled

(i) $\| x, y \| = 0$ if and only if $\{x, y\}$ is a linear dependent set

(ii) $\| x, y \| = \| y, x \|$ for arbitrary $x, y \in X$

(iii) $\| \alpha x, y \| = |\alpha| \cdot \| x, y \|$ for arbitrary $\alpha \in \Phi$ and for arbitrary $x, y \in X$

(iv) $\| x + x', y \| \leq \| x, y \| + \| x', y \|$, for arbitrary $x, y \in X$,

we call **2-norm**, and $(X^2, \| \bullet, \bullet \|)$ we call **2-normed space**.

Definition 2. Let X is a vector space over the field Φ. The mapping $p: X^2 \to \mathbb{R}_{\geq 0}$ for which the following conditions are fulfilled

(i) $p(x, y) \geq 0$ if and only if $\{x, y\}$ is a linear dependent set

(ii) $p(x, y) = p(y, x)$ for arbitrary $x, y \in X$

(iii) $p(\alpha x,y)=|\alpha|\cdot p(x,y)$ for arbitrary $\alpha\in\Phi$ and for arbitrary $x,y\in X$

(iv) $p(x+x',y)\le p(x,y)+p(x',y)$, for arbitrary $x,y\in X$,

we call **2-semi norm**, and (X^2,p) we call **2-semi normed space**.

It is worth to note here that for any 2-norm the following equation is fulfilled $\|x,y\|=\|x,y+\alpha x\|$, for arbitrary $x,y\in X$ and for arbitrary scalar $\alpha\in\Phi$.

Due to the definition of an n-norm and the definition of an n-semi norm it turned out that, on the set X^2, where X is a vector space over the field Φ (Φ is the field of real numbers or the field of complex numbers), it is convenient to consider additional operations, two of which are partial and one of which is a complete operation, with the aim of making the notation and considerations easier.

Definition 3. Let X be a vector space over the field Φ. The set X^2 together with the operations

$$(x,z)+(y,z)=(x+y,z)$$

$$(z,x)+(z,y)=(z,x+y)$$

$$A(x,y)=A(x,y)^T$$

where $x,y,z\in X$ and $A\in M_2(\Phi)$ is called a **2-vector space** or **2-space**.

Comment. The third operation in the previous definition is a complete operation, and on the right-hand side of the equality is a multiplication of a matrix with a vector.

Definition 4. Let X be a vector space over the field Φ. The functional $\Lambda: X^2\to\Phi$ for which the following conditions hold

(a) $\Lambda(x+y,z)=\Lambda(x,z)+\Lambda(y,z)$, for arbitrary $x,y,z\in X$

(b) $\Lambda(x,y)=-\Lambda(y,x)$, for arbitrary $x,y\in X$

(c) $\Lambda(\alpha x,y)=\alpha\Lambda(x,y)$, for arbitrary $x,y\in X$ and $\alpha\in\Phi$.

is called **2-skew-symmetric linear form**.

It is not hard to prove that the previous definition (Definition 4) is equivalent with the following definition.

Definition 5. Let X be a vector space over the field Φ. The functional $\Lambda: X^2\to\Phi$ for which the following conditions hold

(a) $\Lambda(x+y,z)=\Lambda(x,z)+\Lambda(y,z)$, for arbitrary $x,y,z\in X$

(b) $\Lambda(A(x,y))=(\det A)\Lambda(x,y)$, for arbitrary $x,y\in X$ and $A\in M_2(\Phi)$.

is called skew-symmetric 2-linear form or simply 2-linear functional.

Completely analogously to the definition of 2-linear functional, which is essentially a definition of a skew-symmetric 2-form, the definitions of 2-seminorm and 2-norm are changing and adapting.

Definition 2'. Let X be a vector space over the field Φ. The mapping $p: X^2\to\mathbb{R}$ for which the following conditions hold

(a) $p(x+y,z)\le p(x,z)+p(y,z)$, for every $x,y,z\in X$

(b) $p(A(x,y))=|\det A|\,p(x,y)$, for every $x,y\in X$ and $A\in M_2(\Phi)$.

is called a **2-semi norm** and (X^2, p) is called a **2-semi normed space**.

Definition 6. The mapping $\|\cdot\|: X^n \to \mathbb{R}$, $n \geq 2$ for which it is fulfilled that:

(a) $\| x_1, x_2 \| = 0$ if and only if x_1, x_2 are linear dependant vectors;

(b) $\| A(x_1, x_2) \| = | \det A | \| x_1, x_2 \|$, for all $x_1, x_2 \in X$ and for all $A \in M_2(\Phi)$;

(c) $\| x_1 + x_2, x_3 \| \leq \| x_1, x_3 \| + \| x_2, x_3 \|$, for all $x_1, x_2, x_3 \in X$,

we call **2-norm of the vector space** X **,** and the ordered pair $(X, \|\cdot,\cdot\|)$ we call **2-normed space**.

In this section a special type of subsets from X^2 will be considered separately.

Definition 7. The subset $S, S \subseteq X^2$ which is closed with respect to the operations of the 2-space X^2 is called **2-subspace** of X^2.

Of course in these considerations the following two theorems are important.

Theorem 2. *The intersection of an arbitrary family of 2-subspaces of the 2-vector space* X^2 *is a 2-subspace.*

According to the last theorem, each subset $A \subseteq X^2$ determines a 2-subspace S_A, the smallest 2-subspace of the 2-vector space X^2 which contains the set A. We will call the 2-subspace S_A the 2-subspace **generated by the set** A, and the set A **-the generating set**.

In this matter we will consider a special type of generating sets, i.e. a generating set of the form $M \cup \{(u, v)\}$, where M is a special type of a 2-subspace, and $(u, v) \in X^2$ is arbitrarily given where $\{u, v\}$ is a linearly independent set.

The basic question which we will consider here is whether it is possible to extend a 2-skew-symmetric linear form defined on some types, i.e. classes 2-subspaces to a bigger subspace, in the sense of extension of linear functionals, i.e. of the type of Hanh-Banach.

The main aim if all such considerations is whether we can prove the following theorem or some of its variants.

Theorem 3. *Let* S *be a* 2*-subspace of the* 2*-space* X^2, $\Lambda : S \to \mathbb{R}$ *be* 2*-skew-symmetric linear form, and* $p : X^2 \to \mathbb{R}$ *be a mapping for which*

(a) $p(x + y, z) \leq p(x, z) + p(y, z)$, *for all* $x, y, z \in X$

(b) $p(tx, y) = tp(x, y)$, *for all* $x, y \in X$ *and* $t > 0$.

There exists 2*-skew-symmetric linear form* $\Lambda' : X^2 \to \mathbb{R}$, *such that* $\Lambda' / S = \Lambda$.

Each 2-semi norm satisfies the conditions a) and b) from the previous theorem.

Furthermore, in many parts we may come across a special kind of subset of X^2.One type of them is given in the following definition.

Definition 8. The subset $T, T \subseteq X^2$ is called n**-invariant** if $AT \subseteq T$ for every $A \in M_2(\Phi)$, $\det A = 1$.

The general structure of 2-subspaces is, of course, not simple. The simplest forms of 2-subspaces are the kernel subspaces, loop subspaces, branch subspaces and cyclic subspaces. Those are discussed and described in [6].

Solving the problem presented in the last theorem is of course not simple. An affirmation of that is of course the complex structure of the 2-subspaces of the 2-space X^2. Due to this, we will discuss partial cases of this problem.

In this matter we will look at extension of 2-skew-symmetric form defined on a branch-2-subspace and extension of a 2-skew-symmetric form defined on a finite 2-subspace.

From here on, we will assume that the subset $\{x_1, x_2, ..., x_n, ...\}$ is a linearly independent subset of the vector space X, not excluding the case when it is finite.

Definition 9. Let X be a vector space over the field Φ. The 2-subspace S generated by the subset $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),....,(x_{n-1},x_n)\}$, where $\{x_1,x_2,...,x_n\}$ is linearly independent set, is called **finite branch 2-subspace.**

These 2-subspaces are also called finite branch, i.e. finite branch 2-subspace. In other papers one-sided and two-sided branch 2-subspaces, which are sets that are 2-subspaces generated with set in the form $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),....,(x_{n-1},x_n)\}$, will be also considered.

A detailed description of branch 2-subspaces is given in [7]. That is the content of the theorem that follows.

Theorem 4. *If* M *is a branch* 2 *subspace generated by the set* $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),....,(x_{n-1},x_n)\}$ *where* $\{x_1,x_2,...,x_n\}$ *is a linearly independent set, then* $M=\bigcup\limits_{i=2}^{n-1}\ \bigcup\limits_{a_{i-1},a_{i+1}\in\Phi} L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\times L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)$.

In the following part we will consider extension of a finite branch 2-subspace M with the addition of one element (u,v) as well as extension of a 2-skew-symmetric form $\Lambda : M \to \mathbb{R}$, that is a 2-skew-symmetric form on $\Lambda' : M' \to \mathbb{R}$, where $M' = \langle M \cup \{(u,v)\}\rangle$.

The leading result in the description of the special 2-subspaces such as cyclic branch 2-subspaces, kernel 2-subspaces and loop 2-subspaces is the following lemma:

Lemma. *The subspace generated by the elements* $(x_{i-1},x_i),(x_i,x_{i+1}),(x_{i+1},x_{i+2})$, *where* $\{x_{i-1},x_i,x_{i+1},x_{i+2}\}$ *is a linearly independent set, is*

$$L(b_{i+2}x_{i+2}+b_ix_i,x_{i+1})\times L(b_{i+2}x_{i+2}+b_ix_i,x_{i+1})\cup L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\times L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)$$

The idea for such lemma is because here it seems as if we have put together two branches, i.e.

$$L(b_{i+2}x_{i+2}+b_ix_i,x_{i+1})\times L(b_{i+2}x_{i+2}+b_ix_i,x_{i+1}) \qquad (1)$$

and $$L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\times L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i). \qquad (2)$$

Here, as its 2-subspace appears a set determined with

$$M=\{(A(x_i,x_{i+1})^T \,/\, A\in M_2(\Phi)\}.$$

Addition of elements from (1) and (2) certainly is possible, but the result is always an element which can be considered that belongs in one of these 2-subspaces, i.e. either in (1) or in (2).

In this part, according to the considerations and operations that are subject of this analysis, it is completely clear that the following comment should be taken into consideration and be respected.

Comment. If in some part a scalar should come in front of an ordered pair, then it can be done and that scalar can be multiplied in some other pat in the ordered pair. Such pairs of elements belong in the same class of equivalence and in the same 2- subspace of the 2-vector space.

2. EXTENSION OF A FINITE BRANCH 2-SUBSPACE

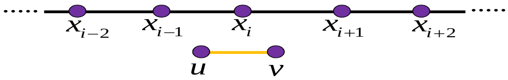
Let Λ be a 2-skew-symmetric linear form defined on a branch 2-subspace M which is generated by the elements of the set $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),....,(x_{m-1},x_m),....\}$, where $\{x_1,x_2,...,x_n,...\}$ is a linearly independent set. Let $(u,v)\in X^2$ be such that $\{u,v\}$ is a linearly independent set. We denote the 2-subspace of X^2 generated by $M\cup\{(u,v)\}$ with M'. Several cases are possible.

Case 1. $u,v\notin L(x_1,x_2,...,x_n)$, where $L(x_1,x_2,...,x_n)$ is the subspace of X generated by $\{x_1,x_2,...,x_n\}$.

The 2-subspace generated by $\{(u,v)\}$ is $L(u,v)\times L(u,v)$. Let us notice that $L(u,v)\cap L(x_1,x_2,...,x_n)=\{0\}(\subseteq\Delta_2)$. Accordingly,

$$M'=M\cup L(u,v)\times L(u,v),$$

where M is determined in theorem 3.



Case 2. Let $u\in L(x_1,x_2,...,x_n)$ and $v\not\in L(x_1,x_2,...,x_n)$.

In this case we will consider several sub cases.

Sub case 1. $u=x_i$ for some $1\le i\le n$.

Situation 1. $u=x_1$

Situation 2. $u=x_n$

Situation 3. $u=x_i$, каде $2\le i\le n-1$

The first and the second sub case are identical and because of that we will consider only the first one.

Situation 1. In this situation we have that $u=x_1$ is an element at the end of the sequence of vectors, so we will have that

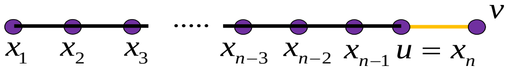
$$v,x_1,x_2,....,x_n=u$$

is a sequence of $n+1$ vectors which as in the case when it is a beginning element of n vectors, form a branch 2- subspace which is a finite branch. Now, the extension of the beginning 2-subspace is given with

$$M' = L(u,v)\times L(u,v)\cup\left(\bigcup_{i=2}^{n-1}\bigcup_{a_{i-1},a_{i+1}\in\Phi} L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\times L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\right)=$$

$$= L(u,v)\times L(u,v)\cup M$$

Situation 3. $u=x_i$, where $2\le i\le n-1$



In this sub case the set $\{(x_{i-1},x_i),(x_i,x_{i+1}),(x_i,v)\}=\{(x_{i-1},u),(u,x_{i+1}),(u,v)\}$ generates a 2-subspace which is a loop subspace and its form is

$$L=\bigcup_{w\in L(x_{i-1},v,x_{i+1})} L(u,w)\times L(u,w).$$

Simultaneously the sets $P'=\{(x_1,x_2),(x_2,x_3),....,(x_{i-2},x_{i-1})\}$ and $P''=\{(x_{i+1},x_{i+2}),(x_{i+2},x_{i+3}),...,(x_{n-1},x_n)\}$ generate 2-subspaces $S_{P'}$ and $S_{P''}$ respectively, which are finite branch 2-subspaces. At the same time, they, as well as L, are 2-subspaces from the required extension M'. The forms of $S_{P'}$ and $S_{P''}$ are

$$S_{P'}=\bigcup_{k=2}^{i-1}\bigcup_{a_{k-1},a_{k+1}\in\Phi} L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)\times L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)$$

$$S_{P''}=\bigcup_{k=i+1}^{n-1}\bigcup_{a_{k-1},a_{k+1}\in\Phi} L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)\times L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)$$

In order for us to see the form of M' it is enough to consider several types of addition of elements of $L, S_{P'}$ and $S_{P''}$. It is enough to consider the cases:

1° $(m,n)\in L$, $(x,y)\in L((x_{i-2},x_{i-1}),(x_{i-1},x_i))$

2° $(m,n)\in L$, $(x,y)\in L((x_{i-3},x_{i-2}),(x_{i-2},x_{i-1}))$

3° $(m,n)\in L$, $(x,y)\in L((x_i,x_{i+1}),(x_{i+1},x_{i+2}))$

4° $(m,n)\in L$, $(x,y)\in L((x_{i+1},x_{i+2}),(x_{i+2},x_{i+3}))$.

In case 1° we have

$$(m,n)=(b_1(\alpha_1x_{i-1}+\alpha_2v+\alpha_3x_{i+1})+b_2x_i, b_3(\alpha_1x_{i-1}+\alpha_2v+\alpha_3x_{i+1})+b_4x_i)$$

$$(x,y)=(a_1(\alpha x_{i-2}+\beta x_i)+a_2x_{i-1}, a_3(\alpha x_{i-2}+\beta x_i)+a_4x_{i-1}).$$

The sum of two such elements is possible in 2 cases:

a) $\alpha_2=\alpha_3=\alpha=0$, $b_1\alpha_1=a_2=s$, $a_1\beta=b_2=t$

b) $\alpha_2=\alpha_3=\alpha=0$, $b_3\alpha_1=a_4=s$, $a_3\beta=b_4=t$

In case a) the elements get the form

$$(b_1\alpha_1x_{i-1}+b_2x_i, b_3\alpha_1x_{i-1}+b_4x_i)=(sx_{i-1}+tx_i, b_3\alpha_1x_{i-1}+b_4x_i)$$

$$(a_1\beta x_i+a_2x_{i-1}, a_3\beta x_i+a_4x_{i-1})=(sx_{i-1}+tx_i, a_3\beta x_i+a_4x_{i-1}),$$

and their sum is $(sx_{i-1}+tx_i,(a_3\beta+b_4)x_i+(a_4+b_3\alpha_1)x_{i-1})\in L((x_{i-1},x_i))\subset L$

We similarly get for case b)

In case 2° we have

$$(x,y)=(a_1(\alpha x_{i-3}+\beta x_{i-1})+a_2x_{i-2},a_3(\alpha x_{i-3}+\beta x_{i-1})+a_4x_{i-2})$$

$$(m,n)=(b_1(\alpha_1x_{i-1}+\alpha_2v+\alpha_3x_{i+1})+b_2x_i,b_3(\alpha_1x_{i-1}+\alpha_2v+\alpha_3x_{i+1})+b_4x_i)$$

The sum of two such elements is possible in 2 cases:

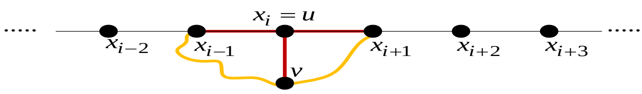
c) $\alpha_2=\alpha_3=\alpha=0$, $a_2=b_2=0$, $a_1\beta=b_1\alpha_1=s$

d) $\alpha_2=\alpha_3=\alpha=0$, $a_4=b_4=0$, $a_3\beta=b_3\alpha_1=s$

In case c) the elements get the form

$$(sx_{i-1},a_3\beta x_{i-1}+a_4x_{i-2})$$

$$(sx_{i-1},b_3\alpha_1x_{i-1}+b_4x_i)$$



and their sum is

$$(sx_{i-1},(a_3\beta+b_3\alpha_1)x_{i-1}+a_4x_{i-2}+b_4x_i)\in L((x_{i-2},x_{i-1}),(x_{i-1},x_i))\subset M$$

We similarly get for case d)

According to that, in this sub case the extension is

$$M'=M\cup\bigcup_{w\in L(x_{i-1},x_i,x_{i+1})}L(x_i,w)\times L(x_i,w).$$

Sub case 2. $u\in L(x_j,x_{j+1})$ for some $j\in\{1,2,...,n-1\}$, where $u\neq x_j,x_{j+1}$.

In other words, we have here a situation in which $u=\alpha_jx_j+\alpha_{j+1}x_{j+1}$, $\alpha_j\alpha_{j+1}\neq 0$. In this sub case we have $u=\mu x_j+\nu x_{j+1}$, where $\mu,\nu\neq 0$. The sets $\{v,u,x_j\}$ and $\{v,u,x_{j+1}\}$ are linearly independent sets. The sets $K'=\{(u,v),(u,x_j)\}$ and $K'=\{(u,v),(u,x_{j+1})\}$ generate 2-subspaces $S_{K'}$ and $S_{K''}$ and their forms are

$$S_{K'}=\bigcup_{\alpha,\beta\in\Phi}L(\alpha v+\beta x_j,u)\times L(\alpha v+\beta x_j,u)$$

$$S_{K''}=\bigcup_{\alpha,\beta\in\Phi}L(\alpha v+\beta x_{j+1},u)\times L(\alpha v+\beta x_j,u)$$

The general form of the elements of $S_{K'}$ is

$$(a_1(\alpha v+\beta x_j)+a_2u,a_3(\alpha v+\beta x_j)+a_4u)$$

and of the elements of $S_{K''}$ is

$$(b_1(\gamma v+\delta x_{j+1})+b_2u,b_3(\gamma v+\delta x_{j+1})+b_4u).$$

Addition of the latter two forms of elements is possible in the following 2 cases:

a) $\beta=\delta=0$, $a_2=b_2=t$, $a_1\alpha=b_1\gamma=s$

b) $\beta=\delta=0$, $a_2=b_2=t$, $a_3\alpha=b_3\gamma=s$.

In case a) the elements get the form

$$(sv+tu,a_3\alpha v+a_4u)$$

$$(sv+tu,b_3\gamma v+b_4u)$$

and their sum is

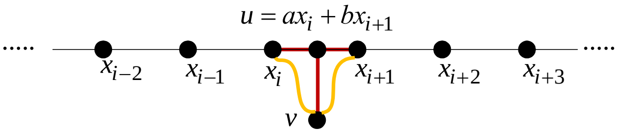
$$(sv+tu,(b_3\gamma+a_3\alpha)v+(a_4+b_4)u)\in L((u,v))\subset M'$$

The result in case b) is similar.

From the whole of the former discussion it is clear that

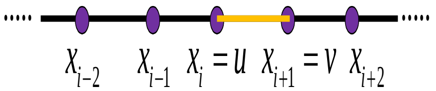
$M' = M \cup S_{K'} \cup S_{K''}$.

We consider the sub cases 3 and 4 similarly.



In this case are considered also the special cases, i.e. when $i = 1$ and when $i + 1 = n$.

Sub case 3. $u \in L(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}, x_i)$ **,** $v \notin L(x_1, x_2, ..., x_{n-1}, x_n)$.



In this situation we have a case when two elements belong in the 2-subspace generated by the elements $(u, x_i), (u, v)$. This 2-subspace can be treated as a branch 2-subspace generated by x_i and v, or we can treat it as a 2-subspace generated by the two dimensional space $L(x_i, v)$. According to this

$$M' = M \cup \bigcup_{w \in L(x_i, v)} L(w,u) \times L(w,u).$$

Here, we will consider also the special cases, i.e. when one of the scalars at u is zero, i.e. $\alpha_{i-1} = 0 \vee \alpha_{i+1} = 0$.

Position 1. $\alpha_{i-1} = 0$

In this situation we have that the vector u has the form $u = \alpha_i x_i + \alpha_{i+1} x_{i+1}$ where we have completely analogue situation as in the sub case 2 of this case, which is completely described.

Position 2. $\alpha_{i+1} = 0$

In this situation we have that the vector u has the form $u = \alpha_{i-1} x_{i-1} + \alpha_i x_i$ where we have completely analogue situation as in the sub case 2 of this case, which is completely described.

Sub case 4. $u \in L(x_i, ..., x_k)$, where $k > i + 2$ and the coefficients in the representation before x_i and x_k are different from zero. In this situation we have that for example k can be at least $i + 3$. In that situation u is not a coordinate vector of an 2-vector element which belongs in M . According to this, this is a case when for u we can consider that it is not in $L(x_1, x_2, ..., x_n)$. The extension here is the same as in the case 1.

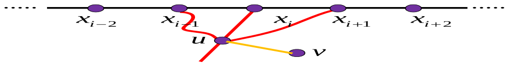
The case $u \notin L(x_1, x_2, ..., x_n)$ and $v \in L(x_1, x_2, ..., x_n)$ is completely analogously considered.

Case 3. Let $u, v \in L(x_1, x_2, ..., x_n)$.

We will consider several possibilities, i.e. sub cases.

Sub case 1. $u = x_i$, $v = x_{i+1}$, including also $i = 1$ or $i = n - 1$

In this sub case $L(u, v) = L(x_j, x_{j+1})$, therefore we don't have a true extension of M . In other words $M = M'$, i.e. we do not have an extension.



Sub case 2. $u = x_i$, $v = x_{i+2}$, including also $i = 1$ and $i = n-2$.

In this sub case, the pairs (x_i, x_{i+1}), (x_{i+1}, x_{i+2}) and (x_i, x_{i+2}) are included in the generating of M' so, accordingly, they define a kernel subspace S which is of the form $L(x_i, x_{i+1}, x_{i+2}) \times L(x_i, x_{i+1}, x_{i+2})$. Now, the subspace M' is generated by one kernel subspace S, and two branch 2-subspaces, one generated by $(x_1, x_2), (x_2, x_3), ..., (x_{i-2}, x_{i-1}), (x_{i-1}, x_i)$ and the other by $(x_{i+2}, x_{i+3}), (x_{i+3}, x_{i+4}), ..., (x_m, x_{m+1}), (x_{m+1}, x_{m+2}), ...$

The form of S is

$$S = L(x_i, x_{i+1}, x_{i+2}) \times L(x_i, x_{i+1}, x_{i+2}) .$$

The form of the 2-subspace S' is

$$S' = \bigcup_{k=2}^{i-1} \bigcup_{a_{k-1}, a_{k+1} \in \Phi} L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k) \times L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k)$$

The form of the 2-subspace S'' is

$$S'' = \bigcup_{k=i+3}^{n} \bigcup_{a_{k-1}, a_{k+1} \in \Phi} L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k) \times L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k)$$

Let us notice that the addition of elements of S or S' or S'' is again an element of S or S' or S'', respectively. Addition of elements of S' and S'', one from S' and the other from S'', is not possible.

We will determine when addition of elements of S and S' is possible and what is the result of that addition. Every element of S is of the form

$$(a_1x_i + b_1x_{i+1} + c_1x_{i+2}, a_2x_i + b_2x_{i+1} + c_2x_{i+2})$$

and the elements from S' for which addition is possible are of the form

$$(d_1(\alpha x_{i-2} + \beta x_i) + e_1x_{i-1}, d_2(\alpha x_{i-2} + \beta x_i) + e_2x_{i-1}) .$$

Addition in this case is possible in the following two cases:

a) $b_1 = c_1 = 0$, $\alpha = 0$, $d_1\beta = a_1 = s$

b) $b_2 = c_2 = 0$, $\alpha = 0$, $d_2\beta = a_2$.

It is enough to consider the case a). Then the elements obtain the form

$$(sx_i, a_2x_i + b_2x_{i+1} + c_2x_{i+2}), \; (sx_i, d_2\beta x_i + e_2x_{i-1})$$

and their sum is

$$(sx_i, (a_2 + d_2\beta)x_i + b_2x_{i+1} + c_2x_{i+2} + e_2x_{i-1}) .$$

Therefore, the sum of these elements is an element from the 2-subspace T defined by

$$T = \bigcup_{u \in L(x_{i-1}, x_{i+1}, x_{i+2})} L(x_1, u) \times L(x_1, u) .$$

Now it is enough to determine the sum of the elements from the 2-subspace T with the elements of the 2-subspace generated by the elements of the set $\{(x_{i-3}, x_{i-2}), (x_{i-2}, x_{i-1})\}$. The former are of the form

$$\begin{aligned} &A(x_i, \alpha_1x_{i-1} + \alpha_2x_i + \alpha_3x_{i+1} + \alpha_4x_{i+2}) = \\ &= (b_1x_i + b_2(\alpha_1x_{i-1} + \alpha_2x_i + \alpha_3x_{i+1} + \alpha_4x_{i+2}), b_3x_i + b_4(\alpha_1x_{i-1} + \alpha_2x_i + \alpha_3x_{i+1} + \alpha_4x_{i+2})) \end{aligned} \qquad (*)$$

The subspace generated by the set $\{(x_{i-3}, x_{i-2}), (x_{i-2}, x_{i-1})\}$ is

$$\bigcup_{\alpha,\beta\in\Phi} L(\alpha x_{i-3}+\beta x_{i-1}, x_{i-2})\times L(\alpha x_{i-3}+\beta x_{i-1}, x_{i-2}),$$

and its elements are of the form

$$(a_1(\alpha x_{i-3}+\beta x_{i-1})+a_2x_{i-2}, a_3(\alpha x_{i-3}+\beta x_{i-1})+a_4x_{i-2}). \qquad (**)$$

Elements of the form (*) and (**) is feasible in two cases:

c) $b_1=0$, $\alpha_2=\alpha_3=\alpha_4=0$, $\alpha=0$, $a_2=0$, $b_2\alpha_1=a_1\beta=s$

d) $b_3=0$, $\alpha_2=\alpha_3=\alpha_4=0$, $\alpha=0$, $a_4=0$, $b_4\alpha_1=a_3\beta=s$.

In the case c) we have

$$(sx_i, b_3x_i+b_4\alpha_1x_{i-1})$$

$$(sx_i, a_3\beta x_{i-1}+a_4x_{i-2})$$

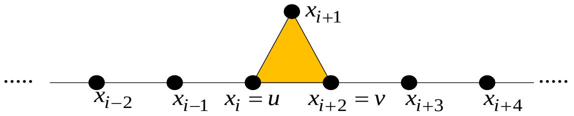
and their sum is

$$(sx_i, b_3x_i+(b_4\alpha_1+a_3\beta)x_{i-1}+a_4x_{i-2})\in L(\gamma x_{i-2}+\delta x_i, x_{i-1})\times L(\gamma x_{i-2}+\delta x_i, x_{i-1}).$$

The case d) is considered analogously.

Accordingly, in this case M' is the 2-subspace

$$M' = M\cup(L(x_i, x_{i+1}, x_{i+2}))^2\cup\bigcup_{u\in L(x_{i-1},x_{i+1},x_{i+2})} L(u,x_i)\times L(u,x_i)\cup\bigcup_{v\in L(x_{i+3},x_{i+1},x_i)} L(v,x_{i+2})\times L(v,x_{i+2}).$$



In the beginning as well in the end situation we would have a loop 2-subspace, at which in the first situation, i.e. $i=1$ is followed by a finite branch 2-subspace, and in the other situation, i.e. $i=n-2$ when a finite branch 2-subspace ends with a loop 2-subspace generated by x_{n-2}, x_{n-1}, x_n.

In other words we would have

a) $M' = M\cup L^2(x_1,x_2,x_3)\cup\bigcup\limits_{w\in L(x_4,x_2,x_1)} L(w,x_3)\times L(w,x_3)$

b) $M' = M\cup L^2(x_{n-2},x_{n-1},x_n)\cup\bigcup\limits_{w\in L(x_{n-3},x_{n-1},x_n)} L(w,x_{n-2})\times L(w,x_{n-2})$

Sub case 3. $u=x_i$, $v=x_j$, $1<i<j<n$, $j>i+2$.

In this sub case the ordered pairs $(x_i,x_{i+1}),(x_{i+1},x_{i+2}),....,(x_{j-1},x_j),(x_j,x_i)$ form a cyclic subspace S. Now, the extension is generated by one cyclic subspace S, and two finite branch 2-subspaces, one S' generated by $(x_1,x_2),(x_2,x_3),....,(x_{i-2},x_{i-1}),(x_{i-1},x_i)$ and the other S'' generated by $(x_j,x_{j+1}),(x_{j+1},x_{j+2}),....,(x_{n-1},x_n)$.

The form of S is.

$$S=\bigcup_{i=1}^{n}\left[L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\times L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\right].$$

The form of S' is

$$S'=\bigcup_{k=2}^{i-1}\bigcup_{a_{k-1},a_{k+1}\in\Phi} L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)\times L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)$$

The form of S'' is

$$S''=\bigcup_{k=j}^{\infty}\ \bigcup_{a_{k-1},a_{k+1}\in\Phi} L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)\times L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)$$

Addition of elements of S' and S'', i.e. one element from S' and the other form S'' is not possible.

We will consider the remaining possibilities for addition of elements of S, S' and S''. Let us notice that the sets $K'=\{(x_{i-1},x_i),(x_i,x_{i+1}),(x_i,x_j)\}$ and $K''=\{(x_i,x_j),(x_{j-1},x_j),(x_j,x_{j+1})\}$ are generators of the 2-subspaces S_K and $S_{K'}$ which are subspaces of M'. At the same time they are loop 2-subspaces generated by three elements. We have:

$$S_{K'}=\bigcup_{u\in L(x_i,x_{j-1},x_{j+1})} L(u,x_j)\times L(u,x_j) \text{ and } S_{K''}=\bigcup_{v\in L(x_{i-1},x_{i+1},x_j)} L(v,x_i)\times L(v,x_i)\ .$$

First we will determine when addition is possible between elements from $S_{K'}$ and $S_{K''}$ and what will the result from the addition be. The elements from $S_{K'}$ are of the form

$$(a_1(\alpha_1x_i+\alpha_2x_{j-1}+\alpha_3x_{j+1})+b_1x_j, a_2(\alpha_1x_i+\alpha_2x_{j-1}+\alpha_3x_{j+1})+b_2x_j)$$

and the elements of $S_{K''}$ are of the form

$$(c_1(\beta_1x_{i-1}+\beta_2x_{i+1}+\beta_3x_j)+d_1x_i, c_2(\beta_1x_{i-1}+\beta_2x_{i+1}+\beta_3x_j)+d_2x_i)\ .$$

It is clear that addition is possible in two cases:

a) $\alpha_2=\alpha_3=\beta_1=\beta_2=0$, $a_1\alpha_1=d_1=s$, $c_1\beta_3=b_1=t$

b) $\alpha_2=\alpha_3=\beta_1=\beta_2=0$, $a_2\alpha_1=d_2=s$, $c_2\beta_3=b_2=t$.

In case a) we have the sum

$$(sx_i+tx_j,(a_2\alpha_1+d_2)x_i+(c_2\beta_3+b_2)x_j)\in L((x_i,x_j))$$

In case b) we have the sum

$$((a_1\alpha_1+d_1)x_i+(c_1\beta_3+b_1)x_j, sx_i+tx_j)\in L((x_i,x_j))$$

Therefore in each case the sum is an element from the 2-subspace $L((x_i,x_j))$

We will determine the sums in the remaining possibilities for addition in M'. We have the following possibilities:

1° $(x,y)\in S_{K''}$ and $(m,n)\in L((x_{i-3},x_{i-2}),(x_{i-2},x_{i-1}))$

2° $(x,y)\in S_{K''}$ and $(m,n)\in L((x_{i+1},x_{i+2}),(x_{i+2},x_{i+3}))$

3° $(x,y)\in S_{K'}$ and $(m,n)\in L((x_{j-3},x_{j-2}),(x_{j-2},x_{j-1}))$

4° $(x,y)\in S_{K'}$ and $(m,n)\in L((x_{j+3},x_{j+2}),(x_{j+2},x_{j+1}))$

In 1° the elements from $S_{K''}$ are of the form

$$(c_1(\beta_1x_{i-1}+\beta_2x_{i+1}+\beta_3x_j)+d_1x_i, c_2(\beta_1x_{i-1}+\beta_2x_{i+1}+\beta_3x_j)+d_2x_i$$

and $(m,n)=(a_1(\alpha x_{i-3}+\beta x_{i-1})+b_1x_{i-2}, a_2(\alpha x_{i-3}+\beta x_{i-1})+b_2x_{i-2}))$. Therefore, addition is possible in the following two cases:

c) $\beta_2=\beta_3=0$, $\alpha=0$, $b_1=0$, $d_1=0$, $c_1\beta_1=a_1\beta=t$

d) $\beta_2=\beta_3=0$, $\alpha=0$, $b_1=0$, $d_1=0$, $c_2\beta_1=a_2\beta=t$

In the case c) we get

$(tx_{i-1}, c_2\beta_1 x_{i-1} + d_2 x_i)$

$(tx_{i-1}, a_2\beta x_{i-1} + b_2 x_{i-2})$

and for the sum we get

$(tx_{i-1}, (c_2\beta_1 + a_2\beta)x_{i-1} + d_2 x_i + b_2 x_{i-2}) \in L((x_{i-2}, x_{i-1}), (x_{i-1}, x_i))$

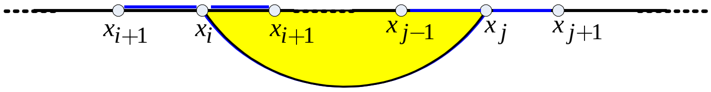
The case d) can be analogously considered.

Similar results are obtained in 2°, 3° and 4° with the results of the additions being elements of the 2-subspaces $L((x_i, x_{i+1}), (x_{i+1}, x_{i+2}))$, $L((x_{j-2}, x_{j-1}), (x_{j-1}, x_j))$ and $L((x_{j+2}, x_{j+1}), (x_{j+1}, x_j))$ respectively, and also being elements of M.

The remaining cases for addition, when it is possible, are addition of elements M and they again belong to M.

Finally, we can conclude that in this sub case:

$$M' = M \cup \bigcup_{u \in L(x_{i-1}, x_{i+1}, x_j)} L(u, x_i) \times L(u, x_i) \cup \bigcup_{v \in L(x_{j+1}, x_i, x_{j-1})} L(v, x_j) \times L(v, x_j).$$

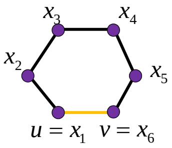


In case when we have exactly four or five generator elements (see drawing), we get that the new 2-subspace will have the form given in the next two positions.

Position 1. If we have 5 elements which generate the starting 2-subspace M then we have the following two situations: $u = x_2, x_3, x_4.x_5 = v$ generate cyclic 2-subspace, and $u = x_2$ is a center of a loop 2-subspace, or $u = x_1, x_2, x_3.x_4 = v$ form a cyclic 2-subspace, and $v = x_4$ is a centre of a loop 2-subspace. According to this, we will have one cyclic 2-subspace S and one loop 2-subspace S'. So, we have that

$M' = S \cup S'$.

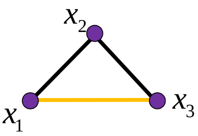
Position 2. In case when we have four generator elements, the case is not like the previous position at all, because then we have that $u = x_1, x_2, x_3.x_4 = v$. According to that, it is made a cyclic 2-subspace generated by the same vectors that are mentioned here.



Sub case 4. $u = x_1$, $v = x_n$

In this situation we get that the ends of the finite branch, i.e. x_1 and x_n now are u and v accordingly. So, the sequence of 2-vectors $(x_1 = u, x_2), (x_2, x_3) ..., (x_{n-1}, x_n)(v = x_n, u)$ generate classical cyclic 2-subspace (see drawing, it is given for case $n = 6$).

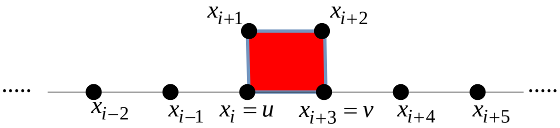
Here it is important to note that characteristic case is the one when the number of generator elements of the starting space is 2, i.e. for the set of linearly independent vectors $\{x_1, x_2, x_3\}$, with the 2-vectors (x_1, x_2) and (x_2, x_3) are



chosen vectors $u = x_1$, $v = x_3$. In that case, the 2-subspace which will be generated is a kernel 2-subspace and that is the only such case (see drawing).

The cases $n = 4, 5, 6, 7, \ldots$, i.e. for all $n \geq 4$ are the same, i.e. they are as illustrated on the drawing.

Sub case 5. $u = x_i$, $v = x_j$ where $j = i + 3 < n$ and $1 < i$



In this situation, we have that the 2-vectors $(u = x_i, x_{i+1}), (x_{i+1}, x_{i+2}), (x_{i+2}, v = x_{i+3}), (v, u = x_i)$ form a four with which is determined one cyclic 2-subspace. But now, the vectors $(v, x_{i+4}), (v, x_{i+2}), (v, u)$ from one side and the three 2-vectors $(u, x_{i-1}), (u, x_{i+1}), (u, v)$, both for itself are loop 2-subspaces, i.e. two loop 2-subspaces. Also we have two branch 2-subspaces, which are finite branch 2-subspaces.

Sub case 6. $u = x_1$, $v = x_{n-1}$, where $n \geq 4$.

In this case we have forming of one cyclic 2-subspace and one loop 2-subspace. This can be seen from the drawing. Here, the cyclic 2-subspace in the case $n > 4$ is fully cyclic 2-subspace as it is shown on the drawing.

In context of the previous discussion, we will have here:

Situation 1. $n = 4$

First, the kernel 2-subspace has the form $M = L(x_1, x_2, x_3) \times L(x_1, x_2, x_3)$ whilst the loop 2-subspace has the form $S = \bigcup_{w \in L(x_1, x_2, x_4)} L(w, v) \times L(w, v)$ and $M' = M \cup S$

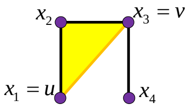
.

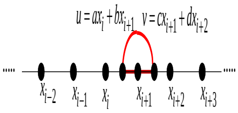
Situation 2. $n \geq 5$

We should note that here it is enough to consider only the case $n = 5$. All other cases are completely analogous. Now,

$$M = \bigcup_{\substack{i=1, i+2>4 \\ i+2\equiv 1 \,(\mathrm{mod}\, 4) \\ i+2\equiv 2 \,(\mathrm{mod}\, 4)}}^{4} \; \bigcup_{\alpha_i, \alpha_{i+2} \in \Phi} L(\alpha_i x_i + \alpha_{i+2} x_{i+2}, x_{i+1}) \times L(\alpha_i x_i + \alpha_{i+2} x_{i+2}, x_{i+1}) \ ,$$

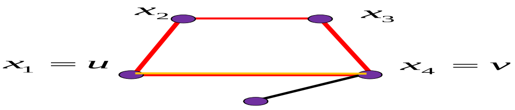
while $S = \bigcup_{w \in L(x_1, x_4, x_3)} L(w, v) \times L(w, v)$. It is clear that $M' = M \cup S$.





Sub case 7. $u = ax_i + bx_{i+1}$, $v = cx_{i+1} + dx_{i+2}$ where $ab \neq 0$ and $cd \neq 0$

In this situation, we have that the three 2-vectors $(u, x_{i+1}), (x_{i+1}, v), (v, u)$ which are determined, determine three vectors v, u, x_{i+1} which are linearly independent.



They determine a kernel 2-subspace which is fully contained in the extended subspace M'.

Procedure 1. The vectors are (u,v) and (u,x_{i+1}). Now, we have that

$$\begin{bmatrix}1&0\\0&\frac{1}{d}\end{bmatrix}\left((u,v)+\begin{bmatrix}1&0\\0&-c\end{bmatrix}(u,x_{i+1})\right)=\begin{bmatrix}1&0\\0&\frac{1}{d}\end{bmatrix}\big((u,cx_{i+1}+dx_{i+2})+(u,-cx_{i+1})\big)=$$

$$=\begin{bmatrix}1&0\\0&\frac{1}{d}\end{bmatrix}(u,dx_{i+2})=(u,x_{i+2})$$

So, we have that the 2-vector $(u,x_{i+2})\in M'$.

Procedure 2. The vectors $(x_{i+2},u),(x_{i+2},x_{i+1})$ are 2-vectors which are from M'. But, now

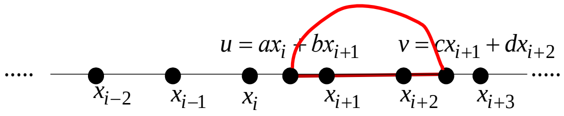
$$\begin{bmatrix}1&0\\0&\frac{1}{a}\end{bmatrix}\left((x_{i+2},u)+\begin{bmatrix}1&0\\0&-b\end{bmatrix}(x_{i+2},x_{i+1})\right)=\begin{bmatrix}1&0\\0&\frac{1}{a}\end{bmatrix}\big((x_{i+2},ax_i+bx_{i+1})+(x_{i+2},-bx_{i+1})\big)=$$

$$=\begin{bmatrix}1&0\\0&\frac{1}{a}\end{bmatrix}(x_{i+2},ax_i)=(x_{i+2},x_i)$$

So, we finally get the 2-vectors $(x_i,x_{i+1}),(x_{i+1},x_{i+2}),(x_{i+2},x_i)$ which are part of the 2-subspace M'. According to this, this 2-subspace is consisted from two 2-subspaces which are finite branch 2-subspaces, one of them is kernel 2-subspace and eventually two loop subspaces. This happens when $i>1$ and $i+2<n$.

In the cases when $i=1$ or $i+2=n$, then we have one starting/ending kernel 2-subspace, one branch less and one loop less.

Sub case 8. $u=ax_i+bx_{i+1}$, $v=cx_{i+2}+dx_{i+3}$, where $ab\neq 0$ and $cd\neq 0$



Here we have a clear picture that the vectors $u=ax_i+bx_{i+1}$ and $v=cx_{i+2}+dx_{i+3}$ are vectors that belong in the current 2-subspace M. But, here in fact is formed one cyclic 2-subspace which is generated by the elements $(u,x_{i+1}),(x_{i+1},x_{i+2}),(x_{i+2},v),(v,u)$, which will be denoted with S. So, here we have additional elements which should be taken in consideration. Certainly, the most interesting part in this context is whether the vectors $u=ax_i+bx_{i+1}$ and $v=cx_{i+2}+dx_{i+3}$ are loops, i.e. if they are loops of two loop 2-subspaces. One of them is generated by $(x_{i+2},v),(v,x_{i+3}),(v,u)$, and the other one is generated by $(x_i,u),(u,x_{i+1}),(u,v)$.

The answer is yes, they are loop 2-subspaces which are generated with loop elements as said, i.e. by $u=ax_i+bx_{i+1}$ and $v=cx_{i+2}+dx_{i+3}$

According to this,

$$M' = M \cup S \cup \bigcup_{w \in L(x_i, x_{i+1}, v)} L(w,u) \times L(w,u) \cup \bigcup_{z \in L(x_{i+2}, x_{i+3}, u)} L(z,v) \times L(z,v)$$

Comment. Similarly as in the part at cyclic 2-subspace generated by elements in this form, if we choose two scalars $\delta, \gamma \in \Phi$, then for them we get that the 2-vector

$$\begin{bmatrix} \delta & 0 \\ 0 & \gamma \end{bmatrix}(u,v) = (\delta u, \gamma v) \in M'.$$

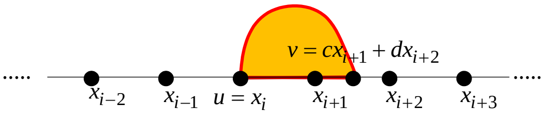
From the other side we have that

$$\begin{bmatrix} \delta & 0 \\ 0 & 1 \end{bmatrix}(u, x_i) = (\delta u, x_i) \in M \subseteq M', \quad \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}(x_{i+2}, v) = (x_{i+2}, \gamma v) \in M \subseteq M'$$

According to this, for the set of four 2-vectors $(\delta u, \gamma v)$, $(\delta u, x_i)$, (x_{i+1}, x_{i+2}), $(x_{i+2}, \gamma v)$ one 2-subspace is generated which is cyclic 2-subspace. That is fulfilled for any $\delta, \gamma \in \Phi$. But, even if they form cyclic 2-subspace, they are a part of already formed cyclic 2-subspace and it doesn't make sense to consider it separately.

Sub case 9. $u = x_i$, $v = ax_{i+1} + bx_{i+2}$, $ab \neq 0$

In this case we have that the three vectors $u = x_i$, x_{i+1}, $v = cx_{i+1} + dx_{i+2}$ are



linearly independent. According to this, the triple of vectors

$$(u = x_i,\ x_{i+1}),\ (x_{i+1}, v = cx_{i+1} + dx_{i+2}), (u = x_i, v = cx_{i+1} + dx_{i+2}),$$

because of the fact that they are all in the newly generated 2-subspace, they form just for themselves a kernel 2-subspace, as from the 2-space X^2, as from the new subspace. Now, the vector x_i which is linearly independent with the vectors x_{i-1}, x_{i+1} and $v = cx_{i+1} + dx_{i+2}$ is a loop center of a loop 2-subspace, which at the same time is also a subspace from X^2 as well as from the new 2-subspace. The problem what happens with $v = cx_{i+1} + dx_{i+2}$ remains, and will it be a loop from the new 2-subspace, which at the same time is a 2-subspace from X^2 as from the new 2-subspace.

The question is whether this kernel 2-subspace can be extended to the kernel 2-subspace in the form $S = L(x_i, x_{i+1}, x_{i+2}) \times L(x_i, x_{i+1}, x_{i+2})$. It is clear that $(x_i, \alpha x_{i+1} + \beta x_{i+2})$ and (x_i, x_{i+1}) are two 2-vectors from the given subspace. Now, it is clear that $(x_i, \alpha x_{i+1} + \beta x_{i+2} - \alpha x_{i+1}) = (x_i, \beta x_{i+2}) \in M$, where from we get that

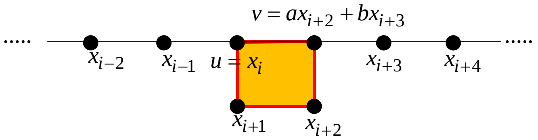
$$\begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{\beta} \end{bmatrix}(x_i, \beta x_{i+2}) = (x_i, x_{i+2}) \in M'.$$

According to this, $(x_i, x_{i+1}), (x_{i+1}, x_{i+2}), (x_{i+2}, x_i)$ are three 2-vectors which are in M', so, the kernel 2-subspace generated from them is consisted in it.

Now, this case overlaps with the sub case 2.

But here, there are many ambiguities regarding what can x_i be? If for example $x_i = x_1$, тthen kernel 2-subspace make the vectors $(x_1, x_2), (x_2, x_3), (x_3, x_1)$ There is only one loop, and that is the vector x_3, and in the continuation there is one branch, and that are the 2-vectors $(x_3, x_4), ..., (x_{n-1}, x_n)$. Also if $x_{i+2} = x_n$, then kernel 2-subspace make the vectors $(x_n, x_{n-1}), (x_{n-1}, x_{n-2}), (x_{n-2}, x_n)$, a finite branch 2-subspace make the vectors $(x_1, x_2), (x_2, x_3), ..., (x_{n-3}, x_{n-2})$ and we have one loop 2-vector, and that is the vector x_{n-2}. Certainly, the previous two comments apply when n is greater or equal than 4, with absent of the finite branch when $n = 4$.

Sub case 10. $u = x_i$, $v = ax_{i+2} + bx_{i+3}$



In this case we have four vectors, x_i, x_{i+1}, x_{i+2}, v which according to the conditions are linearly independent. But, they form four 2-vectors, and they are

$(x_i = u, x_{i+1}), (x_{i+1}, x_{i+2}), (x_{i+2}, v = ax_{i+2} + bx_{i+3}), (v = ax_{i+2} + bx_{i+3}, u)$.

These four 2-vectors form a cyclic 2-subspace when $i = 1$ as well as when $i + 3 = n$. The vector $u = x_i$ is a loop centre when $i > 1$, and the vector $v = ax_{i+2} + bx_{i+3}$ is always a loop centre.

The vectors $x_1, x_2, ..., x_{i-1}$ form a finite branch 2-subspace when $i > 2$.

Sub case 11. $u = ax_i + bx_{i+1}$, $v \in L(ax_{i-1} + bx_{i+1}, x_i)$ **.**

In this sub case are covered all situations when $i = 1$ and when $i + 1 = n$. In this sub case we have that the 2-vectors $(u, x_i), (u, v)$ and (v, x_i) form a kernel 2-subspace. The vector u is at least a loop of a 2-subspace, and so is the vector v.

In this context should be considered all extensions which are considered in the paper [8] for one-sided branch, and that are the sub cases from 11 to 17 from that paper, including all sub cases which have ' sign (ex. 11').

3. EXTENSION OF A 2-SKEW-SYMMETRIC LINEAR FORM

Theorem. *Let* $\Lambda : M \to \mathbb{R}$ *be a* 2*-skew-symmetric form such that* $\Lambda(x, y) \le p(x, y)$ *for every* $(x, y) \in M$, $p : X^2 \to \mathbb{R}$ *be a* 2*-semi norm and* M *is a branch* 2*-subspace of the* 2*-space* X^2. *Let* M' *be an extension of* M *as in sub case 3 of case 2. Then there exists a* 2*-skew-symmetric linear form* $\Lambda' : M' \to \mathbb{R}$ *such that* $\Lambda'/M = \Lambda$

$$-p(-x, y) \le \Lambda(x, y) \le p(x, y) . \qquad (*)$$

Proof. It is enough to consider the case when $\alpha_{i-1} \neq 0$ and $\alpha_{i+1} \neq 0$. According to this, we can consider that the vector u to have the form $u = b_{11}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{12}x_i$, which is first coordinate or second one from the 2-vector $(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i)$. We choose two such vectors, i.e.

$$(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i)$$

$$(u, b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i)\ .$$

According to the conditions of the theorem, we have that

$$\begin{aligned}
&\Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i) + \Lambda(u, b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i) = \\
&\quad = \Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i + b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i) \leq \\
&\quad = p(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i - v + b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i + v) \leq \\
&\quad = p(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i - v) + p(b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i + v)
\end{aligned}$$

So, we get that

$$\begin{aligned}
&\Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1})b_{22}x_i) - p(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i - v) \leq \\
&\quad \leq p(b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i + v) - \Lambda(u, b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i)\ .
\end{aligned}$$

Now, we get that

$$\begin{aligned}
&\sup_{\alpha_{i-1}, \alpha_{i+1}} [\Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i) - p(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i - v)] = d \leq \\
&\quad \leq p(b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i + v) - \Lambda(u, b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i).
\end{aligned}$$

From the last equality it is obvious that

$$\begin{aligned}
&\Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i) - p(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i - v) \leq d \leq \\
&\quad \leq p(b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i + v) - \Lambda(u, b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i).
\end{aligned}$$

i.e. the following inequalities are fulfilled

$$\Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i) - d \leq p(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i - v) \quad (1)$$

$$\Lambda(u, b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i) + d \leq p(b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i + v). \quad (2)$$

Now, we will determine the 2-linear skew-symmetric form $\Lambda' : M' \to \mathbb{R}$ with

$$\Lambda'[A(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i-1}x_{i+1}) + b_{22}x_i + \gamma v)] = (\det A)[\Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i-1}x_{i+1}) + b_{22}x_i) + \gamma d]$$

$$\Lambda'(x, y) = \Lambda(x, y)\ ,\ (x, y) \in M$$

So, $\Lambda'/M = \Lambda$.

From the other side, if in (1) instead α_{i-1} and α_{i+1} we choose $\frac{\alpha_{i-1}}{t}$ and $\frac{\alpha_{i+1}}{t}$, $t > 0$ and if we use the properties of Λ and p accordingly, we get that

$$\Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i) - td \leq p(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i - tv) \ (3)$$

Completely analogous, if in (2) instead α'_{i-1} and α'_{i+1} we choose $\frac{\alpha'_{i-1}}{t}$ and $\frac{\alpha'_{i+1}}{t}$, $t > 0$ accordingly, and if we use the properties of Λ and p, again, we get that

$$\Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i-1}x_{i+1}) + b_{22}x_i) + td \leq p(b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i-1}x_{i+1}) + b_{22}x_i + tv). \quad (4)$$

Now, from (3) and (4) we see that

$$\Lambda'(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i-1}x_{i+1}) + b_{22}x_i + \gamma v) \leq p(b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i-1}x_{i+1}) + b_{22}x_i + \gamma v)$$

where from it is clear that in general case $\Lambda' \leq p$ on M'. In other words, the inequality (*) is fulfilled.

CONFLICT OF INTEREST

No conflict of interest was declared from the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

References

[1] R.Malčeski, A.Malčeski, n*-seminormed space,* Annuaire de l'Institute des Mathématiques, Faculté des Sciences de l'Université "Sv. Kiril et Metodij" - Skopje), 38(1997)

[2] A.Misiak: n*-inner product spaces,* Math.Nachr. 140 (1989)

[3] S.Gähler, *Lineare 2-normierte Raume,* Math.Nach. 28(1965)

[4] A.Malčeski, *Zabeleška za definicijata na 2-normiran prostor,* Mat. Bilten, Tom 26 (2002)

[5] A.Malčeski, V.Manova Erakovik, *Some 2-subspaces of 2-space,* Математички Билтен, 35(LXI), Makedonija, (2007)

[6] D.Mitrinović, *Polinomi I Matrici,* Naučna Knjiga, Beograd (1991)

[7] A.Malčeski, V. Manova Erakovikj. *An extended type of Hahn-Banach for Skew-Symmetric linear forms.* Mat. Bilten, 35(LXI)Tome, 2011, pp 41-49

[8] S.Brsakoska, A.Malčeski, *Extension of one sided branch 2-subspace and some extensions of Hahn-Banach type for skew-symmetric 2-linear functionals defined on it,* Proceedings od CODEMA 2020

[9]A,Malčeski, V. Manova Erakovik, *Algebraic structure of the kernel of the n-semi norm*, Matematički bilten, 31, (LVII), Makedonija, (2007)

[1] Faculty for Natural Sciences and Mathematics, University "Sts. Cyril and Methodius", Skopje, Republic of N. Macedonia
E-mail address: sbrsakoska@gmail.com

[2] Faculty Mechanical Engineering, University "Sts. Cyril and Methodius", Skopje, Republic of N. Macedonia
E-mail address: aleksa.malceski@gmail.com

Publisher
Union od Mathematicians of Macedonia – ARMAGANKA

Editor in chief
Prof. d-r Aleksa Malcheski

CIP - Каталогизација во публикација
Национална и универзитетска библиотека "Св. Климент Охридски", Скопје

51(082)

PROCEEDINGS of CODEMA 2020 = Зборник на трудови од CODEMA 2020. -
Skopje : Armaganka, 2021. - 375 стр. : илустр. ; 25 см

Текст на мак. и англ. јазик. - Фусноти кон текстот. - Библиографија кон трудовите

ISBN 978-608-4904-09-0

а) Математика -- Зборници

COBISS.MK-ID 53570309