Union of Mathematicians of Macedonia - ARMAGANKA

IX SEMINAR OF DIFFERENTIAL EQUATIONS AND ANALYSIS

and

1st CONGRESS OF DIFFERENTIAL EQUATIONS, MATHEMATICAL ANALYSIS AND APPLICATIONS

CODEMA 2020

Proceedings of the CODEMA 2020
Зборник на трудови од CODEMA 2020

Skopje, 2021

EXTENSION OF ONE SIDED BRANCH 2-SUBSPACE AND SOME EXTENSIONS OF HAHN - BANACH TYPE FOR SKEW-SYMMETRIC 2-LINEAR FUNCTIONALS DEFINED ON IT

ISBN 978-608-4904-09-0 **UDC:** 517.982.22:515.173
Slagjana Brsakoska[1], Aleksa Malcheski[2]

Abstract. In this paper 2-subspaces from 2-space X^2, which are from one sided branch 2-subspace type, will be taken in consideration. Then all its possible extensions adding one element (u,v) and their complete description will be considered. Also, all extensions of 2-skew-symmetric linear form defined on 2-subspace M' Hahn-Banach type will be considered, in the cases when one vector belongs in 2-vector from M, and the other does not belong (u belongs and v does not belong and vice versa), as well as cases when the two coordinates of (u,v) do not belong in M.

1. INTRODUCTION

Extensions of mappings is something that is often looked at in various mathematical disciplines. One classical example of extension of a given mapping is of course the Hanh-Banach theorem for linear functionals. One version of it comprises the contents of the following theorem.

Theorem 1. *Let M be a vector subspace of the vector space X. The functional $p: X \to \mathbb{R}$ satisfies the conditions*

a) $p(x+y) \le p(x) + p(y)$

b) $p(tx) = tp(x)$,

for every $x, y \in X$ and $t \ge 0$.

The functional $f: M \to R$ is linear and $f(x) \le p(x)$. There exists a linear functional $\Lambda: X \to \mathbb{R}$ such that $\Lambda / M = f$ and $-p(-x) \le \Lambda(x) \le p(x)$.

Of course, it is worth mentioning here both the definitions, for 2-norm, and especially for 2 semi-norm, which we will use many times further.

Definition 1. Let X be a vector space over the field Φ. The mapping $\| \bullet, \bullet \|: X^2 \to \mathbb{R}_{\ge 0}$ for which the following conditions are fulfilled

(i) $\| x, y \| = 0$ if and only if $\{x, y\}$ is a linear dependent set

(ii) $\| x, y \| = \| y, x \|$ for arbitrary $x, y \in X$

(iii) $\| \alpha x, y \| = |\alpha| \cdot \| x, y \|$ for arbitrary $\alpha \in \Phi$ and for arbitrary $x, y \in X$

(iv) $\| x + x', y \| \le \| x, y \| + \| x', y \|$, for arbitrary $x, y \in X$,

we call **2-norm**, and $(X^2, \| \bullet, \bullet \|)$ we call **2-normed space**.

Definition 2. Let X is a vector space over the field Φ. The mapping $p: X^2 \to \mathbb{R}_{\ge 0}$ for which the following conditions are fulfilled

(i) $p(x, y) \ge 0$ if and only if $\{x, y\}$ is a linear dependent set

(ii) $p(x, y) = p(y, x)$ for arbitrary $x, y \in X$

(iii) $p(\alpha x, y) = |\alpha| \cdot p(x, y)$ for arbitrary $\alpha \in \Phi$ and for arbitrary $x, y \in X$

(iv) $p(x + x', y) \leq p(x, y) + p(x', y)$, for arbitrary $x, y \in X$,

we call **2-semi norm**, and (X^2, p) we call **2-semi normed space**.

It is worth to note here that for any 2-norm the following equation is fulfilled $\| x, y \| = \| x, y + \alpha x \|$, for arbitrary $x, y \in X$ and for arbitrary scalar $\alpha \in \Phi$.

Due to the definition of an n-norm and the definition of an n-semi norm it turned out that, on the set X^2, where X is a vector space over the field Φ (Φ is the field of real numbers or the field of complex numbers), it is convenient to consider additional operations, two of which are partial and one of which is a complete operation, with the aim of making the notation and considerations easier.

Definition 3. Let X be a vector space over the field Φ. The set X^2 together with the operations

$$(x, z) + (y, z) = (x + y, z)$$

$$(z, x) + (z, y) = (z, x + y)$$

$$A(x, y) = A(x, y)^T$$

where $x, y, z \in X$ and $A \in M_2(\Phi)$ is called a **2-vector space** or **2-space**.

Comment. The third operation in the previous definition is a complete operation, and on the right-hand side of the equality is a multiplication of a matrix with a vector.

Definition 4. Let X be a vector space over the field Φ. The functional $\Lambda : X^2 \to \Phi$ for which the following conditions hold

(a) $\Lambda(x + y, z) = \Lambda(x, z) + \Lambda(y, z)$, for arbitrary $x, y, z \in X$

(b) $\Lambda(x, y) = -\Lambda(y, x)$ for arbitrary $x, y \in X$

(c) $\Lambda(\alpha x, y) = \alpha \Lambda(x, y)$, for arbitrary $x, y \in X$ and $\alpha \in \Phi$.

is called **skew-symmetric 2-linear form**.

It is not hard to prove that the previous definition (Definition 4) is equivalent with the following definition.

Definition 5. Let X be a vector space over the field Φ. The functional $\Lambda : X^2 \to \Phi$ for which the following conditions hold

(a) $\Lambda(x + y, z) = \Lambda(x, z) + \Lambda(y, z)$, for arbitrary $x, y, z \in X$

(b) $\Lambda(A(x, y)) = (\det A) \Lambda(x, y)$, for arbitrary $x, y \in X$ and $A \in M_2(\Phi)$

is called **skew-symmetric 2-linear form** or simply **2-linear functional**.

Completely analogously to the definition of 2-linear functional, which is essentially a definition of a 2-skew symmetric form, the definitions of 2-seminorm and 2-norm are changing and adapting.

Definition 2'. Let X be a vector space over the field Φ. The mapping $p : X^2 \to \mathbb{R}$ for which the following conditions hold

(a) $p(x + y, z) \leq p(x, z) + p(y, z)$, for every $x, y, z \in X$

(b) $p(A(x, y)) = |\det A| \, p(x, y)$, for every $x, y \in X$ and $A \in M_2(\Phi)$.

is called a **2-seminorm** and (X^2, p) is called a **2-seminormed space**.

Definition 6. The mapping $\|\cdot\|: X^n \to \mathbb{R}$, $n \geq 2$ for which it is fulfilled that:

(a) $\| x_1, x_2 \| = 0$ if and only if x_1, x_2 are linear dependant vectors;

(b) $\| A(x_1, x_2) \| = |\det A| \| x_1, x_2 \|$, for all $x_1, x_2 \in X$ and for all $A \in M_2(\Phi)$;

(c) $\| x_1 + x_2, x_3 \| \leq \| x_1, x_3 \| + \| x_2, x_3 \|$, for all $x_1, x_2, x_3 \in X$,

we call **2-norm** of the vector space X, and the ordered pair $(X, \|\cdot,\cdot\|)$ we call **2-normed space**.

In this section a special type of subsets from X^2 will be considered separately. In fact, we will consider subsets of X^2 which are from this type.

Definition 7. The subset S, $S \subseteq X^2$ which is closed with respect to the operations of the 2-space X^2 is called **2-subspace** of X^2.

Of course in these considerations the following two theorems are important.

Theorem 2. *The intersection of an arbitrary family of 2-subspaces of the 2-vector space X^2 is a 2-subspace.*

According to the last theorem, each subset $A \subseteq X^2$ determines a 2-subspace S_A, the smallest 2-subspace of the 2-vector space X^2 which contains the set A. We will call the 2-subspace S_A the 2-subspace **generated by the set** A, and the set A **-the generating set**.

In this matter we will consider a special type of generating sets, i.e. a generating set of the form $M \cup \{(u,v)\}$, where M is a special type of a 2-subspace, and $(u,v) \in X^2$ is arbitrarily given where $\{u,v\}$ is a linearly independent set.

The basic question which we will consider here is whether it is possible to extend a 2-skew-symmetric linear form defined on some types, i.e. classes 2-subspaces to a bigger subspace, in the sense of extension of linear functionals, i.e. of the type of Hanh-Banach.

The main aim if all such considerations is whether we can prove the following theorem or some of its variants.

Theorem 3. *Let S be a 2-subspace of the 2-space X^2, $\Lambda : S \to \mathbb{R}$ be 2-skew-symmetric linear form, and $p : X^2 \to \mathbb{R}$ be a mapping for which*

(a) $p(x+y, z) \leq p(x,z) + p(y,z)$, *for all* $x, y, z \in X$

(b) $p(tx, y) = tp(x,y)$, *for all* $x, y \in X$ *and* $t > 0$.

There exists 2-skew-symmetric linear form $\Lambda' : X^2 \to \mathbb{R}$ *, such that* $\Lambda'/S = \Lambda$.

Each 2-seminorm satisfies the conditions a) and b) from the previous theorem.

Furthermore, in many parts we may come across a special kind of subset of X^2. One type of them is given in the following definition.

Definition 8. The subset T, $T \subseteq X^2$ is called n**-invariant** if $AT \subseteq T$ for every $A \in M_2(\Phi)$, $\det A = 1$.

The general structure of 2-subspaces is, of course, not simple. The simplest forms of 2-subspaces are the kernel subspaces, loop subspaces, branch subspaces and cyclic subspaces. Those are discussed and described in [6,7].

Solving the problem presented in the last theorem is of course not simple. An affirmation of that is of course the complex structure of the2-subspaces of the 2-space X^2. Due to this, we will discuss partial cases of this problem.

In this matter we will look at extension of 2-skew-symmetric form defined on a branch-2-subspace and extension of a 2-skew-symmetric form defined on a cyclic2-subspace.

From here on, we will assume that the subset $\{x_1, x_2, ..., x_n, ...\}$ is a linearly independent subset of the vector space X, not excluding the case when it is finite.

Definition 9. Let X be a vector space over the field Φ. The 2-subspace S generated by the subset $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),....,(x_{n-1},x_n),....\}$, where $\{x_1, x_2, ..., x_n, ...\}$ is linearly independent set is called a **one-sided branch 2-subspace.**

These 2-subspaces are also called one sided branches, i.e. one sided branch 2-subspaces. In other papers two-sided branch 2-subspaces, which are sets that are 2-subspaces generated with set in the form $\{...,(x_{-n},x_{-(n-1)}),...,(x_{-1},x_{\circ}),(x_{\circ},x_1),(x_1,x_2),(x_2,x_3),...,(x_{n-1},x_n),....\}$, will be also considered. Parallel to this we can consider also 2-subspaces which are generated with finite number of elements $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),....,(x_{n-1},x_n)\}$.

A detailed description of branch 2-subspaces is given in [7]. That is the content of the theorem that follows.

Theorem 4. *If M is a branch 2-subspace generated by the set $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),....,(x_{m-1},x_m),....\}$ where $\{x_1, x_2, ..., x_n, ...\}$ is a linearly independent set, then*

$$M = \bigcup_{i\in\mathbb{N}\setminus\{1\}} \bigcup_{a_{i-1},a_{i+1}\in\Phi} L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i).$$

In the following part we will consider extension of a branch 2-subspace M with the addition of one element (u,v) as well as extension of a 2-skew-symmetric form $\Lambda : M \to \mathbb{R}$ to a skew-symmetric form on $\Lambda' : M' \to \mathbb{R}$, where $M' = \langle M \cup \{(u,v)\}\rangle$

The leading result in the description of the special 2-subspaces such as cyclic, branch 2-subspaces, kernel 2-subspaces and loop 2-subspaces is the following lemma:

Lemma. *The subspace generated by the elements* $(x_{i-1},x_i),(x_i,x_{i+1}),(x_{i+1},x_{i+2})$, *where* $\{x_{i-1},x_i,x_{i+1},x_{i+2}\}$ *is a linearly independent set is*

$$L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \times L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \cup L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i)$$

The idea for such lemma is because here it seems as if we have put together two branches, i.e.

$$L(b_{i+2}x_{i+2}+b_ix_i,x_{i+1})\times L(b_{i+2}x_{i+2}+b_ix_i,x_{i+1}) \qquad (1)$$

and $L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\times L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)$. (2)

Here, as its 2-subspace appears a set determined with

$$M=\{(A(x_i,x_{i+1})^T \,/\, A\in M_2(\Phi)\}.$$

Addition of elements from (1) and (2) certainly is possible, but the result is always an element which can be considered that belongs in one of these 2-subspaces, i.e. either in (1) or in (2). If it belongs in both sunspaces, then it is an element from the 2-subspace $M=\{(A(x_i,x_{i+1})^T \,/\, A\in M_2(\Phi)\}$. That fact will appear in the whole paper.

2. EXTENSION OF A ONE-SIDED BRANCH 2-SUBSPACE

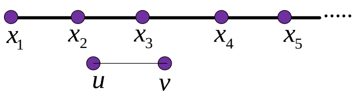
Let Λ be a skew-symmetric linear form defined on a branch 2-subspace M which is generated by the elements of the set $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),....,(x_{m-1},x_m),....\}$, where $\{x_1,x_2,...,x_n,...\}$ is a linearly independent set. Let $(u,v)\in X^2$ be such that $\{u,v\}$ is a linearly independent set. We denote the 2-subspace of X^2 generated by $M\cup\{(u,v)\}$ by M'. Several cases are possible.

Case 1. $u,v\notin L(x_1,x_2,...,x_n,...)$, where $L(x_1,x_2,...,x_n,...)$ is the subspace of X generated by $\{x_1,x_2,...,x_n,...\}$.

The 2-subspace generated by $\{(u,v)\}$ is $L(u,v)\times L(u,v)$. Let us notice that $L(u,v)\cap L(x_1,x_2,...,x_n,...)=\{0\}\subset\Delta_2$. Accordingly,

$$M'=M\cup L(u,v)\times L(u,v),$$

where M is determined in theorem 3.



Case 2. Let $u\in L(x_1,x_2,...,x_n,...)$ and $v\notin L(x_1,x_2,...,x_n,...)$.

In this case we will consider several sub cases.

Sub case 1. $u=x_i$ for some $i\in\mathbb{N}$, and $v\notin L(x_1,x_2,...,x_n,...)$.

In this sub case there are two situations, i.e. $u=x_1$ or $u=x_i$, $i>1$. These situations will be considered separately.

Situation i) $u=x_i$ for some $i\in\mathbb{N}, i>1$

In this situation from sub case 1, the set

$$\{(x_{i-1},x_i),(x_i,x_{i+1}),(x_i,v)\}=\{(x_{i-1},u),(u,x_{i+1}),(u,v)\}$$

generates a 2-subspace which is a loop subspace and its form is

$$L=\bigcup_{w\in L(x_{i-1},v,x_{i+1})} L(u,w)\times L(u,w),$$

even when $i=2$. Now the proof is as follows.

Simultaneously the sets $P'=\{(x_1,x_2),(x_2,x_3),....,(x_{i-2},x_{i-1})\}$ and $P''=\{(x_{i+1},x_{i+2}),(x_{i+2},x_{i+3}),...,(x_{m-1},x_m),...\}$ generate 2-subspaces $S_{P'}$ and $S_{P''}$ respectively, which are branch 2-subspaces. Here, one of them is finite branch 2-subspace, and the other is infinite branch, as it is the starting branch. We should note here that when $i=2$, the 2-subspace P' doesn't exist, and we consider only the 2-subspace P''. But, we will continue with the second case when P' exists. At the same time, they, as well as L, are 2-subspaces from the required extension M'. The forms of $S_{P'}$ and $S_{P''}$ are

$$S_{P'}=\bigcup_{k=2}^{i-1}\ \bigcup_{a_{k-1},a_{k+1}\in\Phi} L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)\times L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)$$

$$S_{P''}=\bigcup_{k=i+1}^{\infty}\ \bigcup_{a_{k-1},a_{k+1}\in\Phi} L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)\times L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)$$

In order for us to see the form of M' it is enough to consider several types of addition of elements of $L,S_{P'}$ and $S_{P''}$ i.e. the following cases:

1° $(m,n)\in L$, $(x,y)\in L((x_{i-2},x_{i-1}),(x_{i-1},x_i))$

2° $(m,n)\in L$, $(x,y)\in L((x_{i-3},x_{i-2}),(x_{i-2},x_{i-1}))$

3° $(m,n)\in L$, $(x,y)\in L((x_i,x_{i+1}),(x_{i+1},x_{i+2}))$

4° $(m,n)\in L$, $(x,y)\in L((x_{i+1},x_{i+2}),(x_{i+2},x_{i+3}))$.

In situation 1° we have

$$(m,n)=(b_1(\alpha_1x_{i-1}+\alpha_2v+\alpha_3x_{i+1})+b_2x_i,b_3(\alpha_1x_{i-1}+\alpha_2v+\alpha_3x_{i+1})+b_4x_i)$$
$$(x,y)=(a_1(\alpha x_{i-2}+\beta x_i)+a_2x_{i-1},a_3(\alpha x_{i-2}+\beta x_i)+a_4x_{i-1}).$$

The sum of two such elements is possible in 2 cases:

a) $\alpha_2=\alpha_3=\alpha=0$, $b_1\alpha_1=a_2=s$, $a_1\beta=b_2=t$

b) $\alpha_2=\alpha_3=\alpha=0$, $b_3\alpha_1=a_4=s$, $a_3\beta=b_4=t$

In case a) the elements get the form

$$(b_1\alpha_1x_{i-1}+b_2x_i,b_3\alpha_1x_{i-1}+b_4x_i)=(sx_{i-1}+tx_i,b_3\alpha_1x_{i-1}+b_4x_i)$$
$$(a_1\beta x_i+a_2x_{i-1},a_3\beta x_i+a_4x_{i-1})=(sx_{i-1}+tx_i,a_3\beta x_i+a_4x_{i-1}),$$

and their sum is

$$(sx_{i-1}+tx_i,(a_3\beta+b_4)x_i+(a_4+b_3\alpha_1)x_{i-1})\in L((x_{i-1},x_i))\subset L$$

We similarly get for case b).

In case 2° we have

$$(x,y)=(a_1(\alpha x_{i-3}+\beta x_{i-1})+a_2x_{i-2},a_3(\alpha x_{i-3}+\beta x_{i-1})+a_4x_{i-2})$$
$$(m,n)=(b_1(\alpha_1x_{i-1}+\alpha_2v+\alpha_3x_{i+1})+b_2x_i,b_3(\alpha_1x_{i-1}+\alpha_2v+\alpha_3x_{i+1})+b_4x_i)$$

The sum of two such elements is possible in 2 cases:

c) $\alpha_2=\alpha_3=\alpha=0$, $a_2=b_2=0$, $a_1\beta=b_1\alpha_1=s$

d) $\alpha_2=\alpha_3=\alpha=0$, $a_4=b_4=0$, $a_3\beta=b_3\alpha_1=s$

In case c) the elements get the form

$$(sx_{i-1},a_3\beta x_{i-1}+a_4x_{i-2})$$

$(sx_{i-1}, b_3\alpha_1 x_{i-1} + b_4 x_i)$

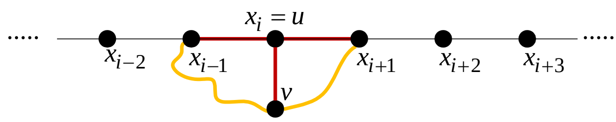
and their sum is

$(sx_{i-1}, (a_3\beta + b_3\alpha_1)x_{i-1} + a_4 x_{i-2} + b_4 x_i) \in L((x_{i-2}, x_{i-1}), (x_{i-1}, x_i)) \subset M$

We similarly get for case d).

According to that, in this sub case the extension is

$$M' = M \cup \bigcup_{w \in L(x_{i-1}, x_i, x_{i+1})} L(x_i, w) \times L(x_i, w) .$$



Situation ii) $i = 1$.

In this situation, together with the condition that v is an element which doesn't belong as coordinate in any of the elements in M, i.e. $v \notin L(x_1, x_2, ...)$, we get that the 2-subspace M is extended and it is again a branch 2-subspace from X^2. In fact, that is a branch determined with the set $v, x_1, x_2, ..., x_n, ...$, which is not hard to describe.

Sub case 2. $u \in L(x_j, x_{j+1})$ for some $j \in \mathbb{N}$, where $u \neq x_j, x_{j+1}$.

Here, maybe it is more convenient to consider that $u = \alpha_j x_j + \alpha_{j+1} x_{j+1} \in L(x_j, x_{j+1})$, and $v \notin L(x_1, x_2, ...)$.

In this sub case we have $u = \mu x_j + \nu x_{j+1}$, where $\mu, \nu \neq 0$. The sets $\{v, u, x_j\}$ and $\{v, u, x_{j+1}\}$ are linearly independent sets. The sets $K' = \{(u, v), (u, x_j)\}$ and $K' = \{(u, v), (u, x_{j+1})\}$ generate 2-subspaces $S_{K'}$ and $S_{K''}$ and their forms are

$$S_{K'} = \bigcup_{\alpha, \beta \in \Phi} L(\alpha v + \beta x_j, u) \times L(\alpha v + \beta x_j, u)$$

$$S_{K''} = \bigcup_{\alpha, \beta \in \Phi} L(\alpha v + \beta x_{j+1}, u) \times L(\alpha v + \beta x_j, u)$$

The general form of the elements of $S_{K'}$ is

$(a_1(\alpha v + \beta x_j) + a_2 u, a_3(\alpha v + \beta x_j) + a_4 u)$

and of the elements of $S_{K''}$ is

$(a_1(\gamma v + \delta x_{j+1}) + a_2 u, a_3(\gamma v + \delta x_{j+1}) + a_4 u)$.

Addition of the latter two forms of elements is possible in the following 2 cases:

a) $\beta = \delta = 0$, $a_2 = b_2 = t$, $a_1\alpha = b_1\gamma = s$

b) $\beta = \delta = 0$, $a_2 = b_2 = t$, $a_3\alpha = b_3\gamma = s$.

In case a) the elements get the form

$(sv + tu, a_3\alpha v + a_4 u)$

$(sv + tu, b_3\gamma v + b_4 u)$

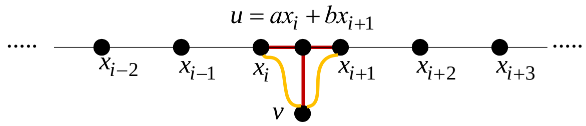
and their sum is

$(sv + tu, (b_3\gamma + a_3\alpha)v + (a_4 + b_4)u) \in L((u, v)) \subset M'$

The result in case b) is similar.

From the whole of the former discussion it is clear that

$M' = M \cup S_{K'} \cup S_{K''}$.



We consider the sub cases 3 and 4 similarly.

Sub case 3. $u \in L(x_1,....,x_k)$**,** $k > 3$**,** $u = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + ... + \alpha_k x_k$**,** $\alpha_1 \alpha_k \neq 0$.

In this sub case, the vector u is not a coordinate of any of the 2-vectors in M, so the extension in this sub case is the same as in the case 1.

Sub case 4. $u \in L(x_j,....,x_k)$**,** $k \geq i+3$**,**

$u = \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2} + \alpha_{i+3} x_{i+3} + \alpha_{i+4} x_{i+4} + ...$**,** $\alpha_i \alpha_{i+3} \neq 0$.

In this sub case, same as in the previous sub case, the vector u is not a coordinate of any of the 2-vectors in M, so the extension in this sub case is the same as in the case 1.

The case $u \notin L(x_1, x_2,..., x_n,...)$ and $v \in L(x_1, x_2,..., x_n,...)$ is completely analogously considered.

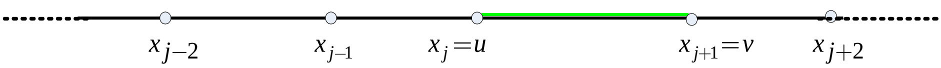
Case 3. Let $u, v \in L(x_1, x_2,..., x_n,...)$.

We will consider several possibilities, i.e. sub cases.

Sub case 1. $u = x_i$, $v = x_{i+1}$.

In this situation in completely analogous way are considered also the case $i = 1$ and all other cases for $i > 1$.

In this sub case $L(u,v) = L(x_j, x_{j+1})$, therefore we don't have a true extension of M.



The same is the discussion when the 2-subspace begins with the element x_1. In this case, the vector (u,v) is in fact the vector (x_1, x_2).

Sub case 2. $u = x_i$, $v = x_{i+2}$, $i > 1$

In this sub case, the pairs (x_i, x_{i+1}), (x_{i+1}, x_{i+2}) and (x_i, x_{i+2}) are included in the generating of M' so, accordingly, they define a kernel subspace S which is of the form $L(x_i, x_{i+1}, x_{i+2}) \times L(x_i, x_{i+1}, x_{i+2})$. Now, the subspace M' is generated by one kernel subspace S, and two branch 2-subspaces, one generated by $(x_1,x_2),(x_2,x_3),....,(x_{i-2},x_{i-1}),(x_{i-1},x_i)$ and the other by $(x_{i+2},x_{i+3}),(x_{i+3},x_{i+4}),....,(x_m,x_{m+1}),(x_{m+1},x_{m+2}),....$

The form of S is

$S = L(x_i, x_{i+1}, x_{i+2}) \times L(x_i, x_{i+1}, x_{i+2})$.

The form of the 2-subspace S' is

$$S'=\bigcup_{k=2}^{i-1}\bigcup_{a_{k-1},a_{k+1}\in\Phi}L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)\times L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)$$

The form of the 2-subspace S'' is

$$S''=\bigcup_{k=i+3}^{\infty}\bigcup_{a_{k-1},a_{k+1}\in\Phi}L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)\times L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)$$

Let us notice that the addition of elements of S or S' or S'' is again an element of S or S' or S'', respectively. Addition of elements of S' and S'', one from S' and the other from S'' is not possible.

We will determine when addition of elements of S and S' is possible and what is the result of that addition. Every element of S is of the form

$$(a_1x_i+b_1x_{i+1}+c_1x_{i+2},a_2x_i+b_2x_{i+1}+c_2x_{i+2})$$

and the elements from S' for which addition is possible are of the form

$$(d_1(\alpha x_{i-2}+\beta x_i)+e_1x_{i-1},d_2(\alpha x_{i-2}+\beta x_i)+e_2x_{i-1}).$$

Addition in this case is possible in the following two cases:

a) $b_1=c_1=0$, $\alpha=0$, $d_1\beta=a_1=s$

b) $b_2=c_2=0$, $\alpha=0$, $d_2\beta=a_2$.

It is enough to consider the case a). Then the elements obtain the form

$$(sx_i,a_2x_i+b_2x_{i+1}+c_2x_{i+2}),\ (sx_i,d_2\beta x_i+e_2x_{i-1})$$

and their sum is

$$(sx_i,(a_2+d_2\beta)x_i+b_2x_{i+1}+c_2x_{i+2}+e_2x_{i-1}).$$

Therefore, the sum of these elements is an element from the 2-subspace T defined by

$$T=\bigcup_{u\in L(x_{i-1},x_{i+1},x_{i+2})}L(x_1,u)\times L(x_1,u).$$

Now it is enough to determine the sum of the elements from the 2-subspace T with the elements of the2-subspace generated by the elements of the set $\{(x_{i-3},x_{i-2}),(x_{i-2},x_{i-1})\}$. The former are of the form

$$A(x_i,\alpha_1x_{i-1}+\alpha_2x_i+\alpha_3x_{i+1}+\alpha_4x_{i+2})=$$
$$=(b_1x_i+b_2(\alpha_1x_{i-1}+\alpha_2x_i+\alpha_3x_{i+1}+\alpha_4x_{i+2}),b_3x_i+b_4(\alpha_1x_{i-1}+\alpha_2x_i+\alpha_3x_{i+1}+\alpha_4x_{i+2}))\quad(*)$$

The subspace generated by the set $\{(x_{i-3},x_{i-2}),(x_{i-2},x_{i-1})\}$ is

$$\bigcup_{\alpha,\beta\in\Phi}L(\alpha x_{i-3}+\beta x_{i-1},x_{i-2})\times L(\alpha x_{i-3}+\beta x_{i-1},x_{i-2}),$$

and its elements are of the form

$$(a_1(\alpha x_{i-3}+\beta x_{i-1})+a_2x_{i-2},a_3(\alpha x_{i-3}+\beta x_{i-1})+a_4x_{i-2}).\qquad(**)$$

Elements of the form (*) and (**) is feasible in two cases:

c) $b_1=0$, $\alpha_2=\alpha_3=\alpha_4=0$, $\alpha=0$, $a_2=0$, $b_2\alpha_1=a_1\beta=s$

d) $b_3=0$, $\alpha_2=\alpha_3=\alpha_4=0$, $\alpha=0$, $a_4=0$, $b_4\alpha_1=a_3\beta=s$.

In the case c) we have

$(sx_i, b_3x_i + b_4\alpha_1 x_{i-1})$

$(sx_i, a_3\beta x_{i-1} + a_4 x_{i-2})$

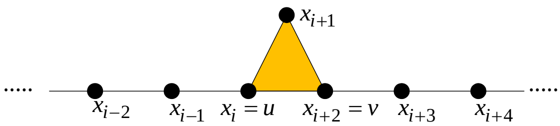
and their sum is

$(sx_i, b_3x_i + (b_4\alpha_1 + a_3\beta)x_{i-1} + a_4x_{i-2}) \in L(\gamma x_{i-2} + \delta x_i, x_{i-1}) \times L(\gamma x_{i-2} + \delta x_i, x_{i-1})$.

The case d) is considered analogously.

Accordingly, in this case M' is the 2-subspace

$$M' = M \cup (L(x_i, x_{i+1}, x_{i+2}))^2 \cup \bigcup_{u \in L(x_{i-1}, x_{i+1}, x_{i+2})} L(u, x_i) \times L(u, x_i) \cup \bigcup_{v \in L(x_{i+3}, x_{i+1}, x_i)} L(v, x_{i+2}) \times L(v, x_{i+2})$$



. **Sub case 2'.** $u = x_1$, $v = x_3$.

In this sub case, the 2-vectors (x_1, x_2), (x_2, x_3) and (x_3, x_1) which are in the new 2-subspace, form a kernel 2-subspace from the form $S = L(x_1, x_2, x_3) \times L(x_1, x_2, x_3)$. On it a branch 2-subspace is added on, with form

$$S' = \bigcup_{k=4}^{\infty} \bigcup_{a_{k-1}, a_{k+1} \in \Phi} L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k) \times L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k) .$$

This new 2-subspace has form

$M' = S \cup S'$

Sub case 3. $u = x_i$, $v = x_j$, $i > 1$ and $j > i + 2$.

This situation when $i > 1$ is similar to the previous one. But now, additionally appears one more loop 2-subspace with loop in the vector x_i, besides the loop x_j which is analogous to the loop in x_4. If $i > 2$, then besides the appearance of one cyclic 2-subspace, two loop 2-subspaces, one branch 2-subspace, appears one more branch 2-subspace which at the same time is a finite branch, generated by the elements $(x_1, x_2), (x_2, x_3), ..., (x_{i-2}, x_{i-1})$. But now let's consider them one by one.

In this sub case the ordered pairs $(x_i, x_{i+1}), (x_{i+1}, x_{i+2}), ..., (x_{j-1}, x_j), (x_j, x_i)$ form a cyclic subspace S. Now, the extension is generated by one cyclic subspace S, and two branch 2-subspaces, one S' generated by $(x_1, x_2), (x_2, x_3), ..., (x_{i-2}, x_{i-1}), (x_{i-1}, x_i)$ and the other S'' generated by $(x_j, x_{j+1}), (x_{j+1}, x_{j+2}), ..., (x_m, x_{m+1}), (x_{m+1}, x_{m+2}), ...$.

The form of S is.

$$S = \bigcup_{i=1}^{n} [L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i)] .$$

The form of S' is

$$S' = \bigcup_{k=2}^{i-1} \bigcup_{a_{k-1}, a_{k+1} \in \Phi} L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k) \times L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k)$$

The form of S'' is

$$S'' = \bigcup_{k=j}^{\infty} \bigcup_{a_{k-1},a_{k+1}\in\Phi} L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k) \times L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k)$$

Addition of elements of S' and S'', i.e. one element from S' and the other form S'' is not possible.

We will consider the remaining possibilities for addition of elements of S, S' and S''. Let us notice that the sets $K' = \{(x_{i-1}, x_i), (x_i, x_{i+1}), (x_i, x_j)\}$ and $K'' = \{(x_i, x_j), (x_{j-1}, x_j), (x_j, x_{j+1})\}$ are generators of the 2-subspaces S_K and $S_{K'}$ which are subspaces of M'. At the same time they are loop 2-subspaces generated by three elements. We have:

$S_{K'} = \bigcup_{u\in L(x_i,x_{j-1},x_{j+1})} L(u, x_j) \times L(u, x_j)$ and $S_{K''} = \bigcup_{v\in L(x_{i-1},x_{i+1},x_j)} L(v, x_i) \times L(v, x_i)$.

First we will determine when addition is possible between elements from $S_{K'}$ and $S_{K''}$ and what will the result from the addition be. The elements from $S_{K'}$ are of the form

$(a_1(\alpha_1 x_i + \alpha_2 x_{j-1} + \alpha_3 x_{j+1}) + b_1 x_j, a_2(\alpha_1 x_i + \alpha_2 x_{j-1} + \alpha_3 x_{j+1}) + b_2 x_j)$

and the elements of $S_{K''}$ are of the form

$(c_1(\beta_1 x_{i-1} + \beta_2 x_{i+1} + \beta_3 x_j) + d_1 x_i, c_2(\beta_1 x_{i-1} + \beta_2 x_{i+1} + \beta_3 x_j) + d_2 x_i)$.

It is clear that addition is possible in two cases:

a) $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = 0$, $a_1\alpha_1 = d_1 = s$, $c_1\beta_3 = b_1 = t$

b) $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = 0$, $a_2\alpha_1 = d_2 = s$, $c_2\beta_3 = b_2 = t$.

In case a) we have the sum

$(sx_i + tx_j, (a_2\alpha_1 + d_2)x_i + (c_2\beta_3 + b_2)x_j) \in L((x_i, x_j))$

In case b) we have the sum

$((a_1\alpha_1 + d_1)x_i + (c_1\beta_3 + b_1)x_j, sx_i + tx_j) \in L((x_i, x_j))$

Therefore in each case the sum is an element from the 2-subspace $L((x_i, x_j))$.

We will determine the sums in the remaining possibilities for addition in M' We have the following possibilities:

1° $(x, y) \in S_{K''}$ and $(m, n) \in L((x_{i-3}, x_{i-2}), (x_{i-2}, x_{i-1}))$

2° $(x, y) \in S_{K''}$ and $(m, n) \in L((x_{i+1}, x_{i+2}), (x_{i+2}, x_{i+3}))$

3° $(x, y) \in S_{K'}$ and $(m, n) \in L((x_{j-3}, x_{j-2}), (x_{j-2}, x_{j-1}))$

4° $(x, y) \in S_{K'}$ and $(m, n) \in L((x_{j+3}, x_{j+2}), (x_{j+2}, x_{j+1}))$

In 1° the elements from $S_{K''}$ are of the form

$(c_1(\beta_1 x_{i-1} + \beta_2 x_{i+1} + \beta_3 x_j) + d_1 x_i, c_2(\beta_1 x_{i-1} + \beta_2 x_{i+1} + \beta_3 x_j) + d_2 x_i$

and $(m, n) = (a_1(\alpha x_{i-3} + \beta x_{i-1}) + b_1 x_{i-2}, a_2(\alpha x_{i-3} + \beta x_{i-1}) + b_2 x_{i-2}))$.

Therefore, addition is possible in the following two cases:

c) $\beta_2 = \beta_3 = 0$, $\alpha = 0$, $b_1 = 0$, $d_1 = 0$, $c_1\beta_1 = a_1\beta = t$

d) $\beta_2 = \beta_3 = 0$, $\alpha = 0$, $b_1 = 0$, $d_1 = 0$, $c_2\beta_1 = a_2\beta = t$

In the case c) we get

$(tx_{i-1}, c_2\beta_1 x_{i-1} + d_2 x_i)$

$(tx_{i-1}, a_2\beta x_{i-1} + b_2 x_{i-2})$

and for the sum we get

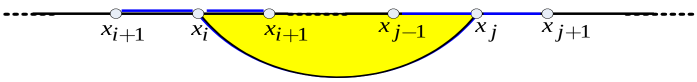
$(tx_{i-1}, (c_2\beta_1 + a_2\beta)x_{i-1} + d_2 x_i + b_2 x_{i-2}) \in L((x_{i-2}, x_{i-1}), (x_{i-1}, x_i))$

The case d) can be analogously considered.

Similar results are obtained in 2°, 3° and 4° with the results of the additions being elements of the 2-subspaces $L((x_i, x_{i+1}), (x_{i+1}, x_{i+2}))$, $L((x_{j-2}, x_{j-1}), (x_{j-1}, x_j))$ and $L((x_{j+2}, x_{j+1}), (x_{j+1}, x_j))$ respectively, and also being elements of M.

The remaining cases for addition, when it is possible, are addition of elements M and they again belong to M.

Finally, we can conclude that in this sub case:



$$M' = M \cup \bigcup_{u \in L(x_{i-1}, x_{i+1}, x_j)} L(u, x_i) \times L(u, x_i) \cup \bigcup_{v \in L(x_{j+1}, x_i, x_{j-1})} L(v, x_j) \times L(v, x_j) .$$

The sub case $u = x_2, v = x_j$, $j > 4$, due to its specifics we will consider it separately.

In this situation we have that the vector x_2 is a loop element, same as the vector $v = x_j$.

Sub case 3'. $u = x_1$, $v = x_j$, $j > 3$.

It is enough to consider the situation when $j = 4$. In this situation, we have 2-vectors $(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_1)$, which belong in the new 2-vector subspace. According to this, they form a cyclic 2-subspace, which we didn't have before. Its form is $S = \bigcup_{i=1}^{4} \bigcup_{a_{i+1}, a_{i-1} \in \Phi} \left[L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \right]$,

which is at the beginning, and then follows a branch 2-subspace generated from already existing 2-vectors $(x_5, x_6), (x_6, x_7), \ldots$. Its form is

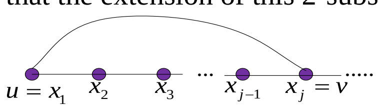
$$S' = \bigcup_{k=5}^{\infty} \bigcup_{a_{k-1}, a_{k+1} \in \Phi} L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k) \times L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k) .$$

But, here appears one loop 2-subspace for which a loop element is the vector x_4. Its form is

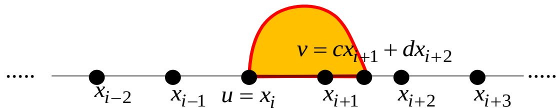
$$S'' = \bigcup_{w \in L(x_1, x_3, x_5)} L(w, x_4) \times L(w, x_4) .$$

Between the elements of these three types of 2-subspaces should be determined addition and we should see what will the results be.

In any case, we have that the extension of this 2-subspace is $M' = S \cup S' \cup S''$



Sub case 4. $u = x_i$, $v = cx_{i+1} + dx_{i+2}$, where $cd \neq 0$ for some $i > 1$



Now, because the 2-vectors $(v,u),(x_{i+1},u) \in M'$, we get that also the 2-vector

$$\begin{bmatrix} \frac{1}{d} & 0 \\ 0 & 1 \end{bmatrix}\left((v,u) + \begin{bmatrix} -c & 0 \\ 0 & 1 \end{bmatrix}(x_{i+1},u)\right) = \begin{bmatrix} \frac{1}{d} & 0 \\ 0 & 1 \end{bmatrix}\left((cx_{i+1} + dx_{i+2}, x_i) + (-cx_{i+1}, x_i)\right) = \begin{bmatrix} \frac{1}{d} & 0 \\ 0 & 1 \end{bmatrix}(dx_{i+2}, x_i) = (x_{i+2}, x_i) \in M'$$

According to this, in this new 2-subspace belong the 2-vectors

$$(u, x_{i+1}) = (x_i, x_{i+1}), (x_{i+1}, x_{i+2}), (x_{i+2}, x_i) = (x_{i+2}, u),$$

and together with that also the kernel subspace generated by the vectors x_i, x_{i+1}, x_{i+2}. Now it is clear that this extension is equal to the extension in the sub case 2 of this case.

Sub case 5. $u = x_1$, $v = ax_2 + bx_3$, $ab \neq 0$.

In this situation the vectors $x_1, x_2, ax_1 + bx_2$ form a triple of vectors which are linearly independent. According to this, the triple of 2-vectors

$$(x_1, x_2) = (u, x_2), (x_2, ax_2 + bx_3) = (x_2, v), (ax_2 + bx_3, x_1) = (v, u),$$

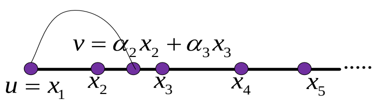
form a kernel 2-subspace in the new 2-vector subspace M'.

But now, since the 2-vectors $(v,u),(x_{i+1},u) \in M'$, we get that also the 2-vector

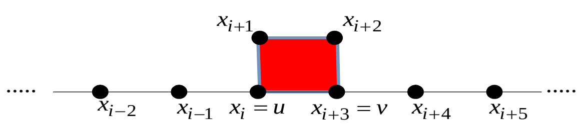
$$\begin{bmatrix} \frac{1}{b} & 0 \\ 0 & 1 \end{bmatrix}\left((v, x_1) + \begin{bmatrix} -a & 0 \\ 0 & 1 \end{bmatrix}(x_2, x_1)\right) = \begin{bmatrix} \frac{1}{b} & 0 \\ 0 & 1 \end{bmatrix}\left((ax_2 + bx_3, x_i) - ax_2, x_1)\right) = \begin{bmatrix} \frac{1}{b} & 0 \\ 0 & 1 \end{bmatrix}(bx_3, x_1) = (x_3, x_1) \in M' \cdot$$

According to this, the 2-vectors $(u, x_2) = (x_1, x_2), (x_2, x_3), (x_3, x_1)$ belong in this new 2-subspace, and with that also the kernel subspace which is generated by the vectors x_1, x_2, x_3. Now it is clear that this extension is equal to the extension from sub case 2' in this case.

Additionally, as an extension of this 2-vector subspace appears the branch 2-subspace which is one sided branch and is generated by the elements $(x_3, x_4), (x_4, x_5), \ldots$. Parallel to this we have vector $v = x_3$ for which we ask the question whether it can be a loop of a loop 2-subspace generated by the kernel 2-subspace which is already generated by the element x_3. Of course the answer is yes, it is a loop of the kernel 2-subspace generated by x_1, x_2, x_3 and the 2-vector (x_3, x_4).



Sub case 6. $u = x_i$, $v = x_j$ where $j = i + 3$.



In this sub case we will consider two situations, i.e.:

a) $i=1$, and b) $i>1$

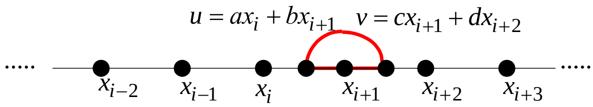
Situation a)

If $i=1$, then we have a cyclic 2-subspace generated by the 2-vectors $(x_1,x_2),(x_2,x_3),(x_3,x_4),(x_4,x_1)$. Then this 2-subspace is followed by loop-connecting 2-subspace with a loop in x_4. It is generated by the 2-vectors $(x_3,x_4),(x_1,x_4),(x_5,x_4)$. Then, from x_5 on, is followed a branch 2-subspace, which is one sided.

Situation b)

This situation is completely analogous to the situation a), and here the role of the vector x_1 takes the vector x_i. Additionally, we have that the vectors $x_1,x_2,....,x_{i-1}$ form a branch 2-subspace, which is a subspace from M'.

Sub case 7. $u=ax_i+bx_{i+1}$, $v=cx_{i+1}+dx_{i+2}$ where $ab\neq 0$ and $cd\neq 0$



In this sub case we have that the 2-vectors (v,u) and (x_{i+1},u) belong in the new 2-subspace M', so, in this 2-subspace belongs also the 2-vector

$$\begin{bmatrix}\frac{1}{d} & 0\\ 0 & 1\end{bmatrix}\left((v,u)+\begin{bmatrix}-c & 0\\ 0 & 1\end{bmatrix}(x_{i+1},u)\right)=\begin{bmatrix}\frac{1}{d} & 0\\ 0 & 1\end{bmatrix}\left((cx_{i+1}+dx_{i+2},u)+(-cx_{i+2},u)\right)=\begin{bmatrix}\frac{1}{d} & 0\\ 0 & 1\end{bmatrix}(dx_{i+2},u)=(x_{i+2},u)$$

Now it is clear that we have 2-subspace which is fully analogous to the 2-subspace which is generated as in sub case 4 from this case, which is equivalent with the sub case 2, and this sub case is fully described. So, the 2-vectors $(x_{i+1},x_{i+2}),(x_{i+2},x_i),(x_i,x_{i+1})$ all belong in M', where from we get that the kernel 2-subspace generated by them is also a subspace from M'. So, the kernel 2-subspace generated by $(x_{i+1},v),(v,u),(u,x_{i+1})$ is consisted both in the kernel 2-subspace generated by $(x_{i+1},x_{i+2}),(x_{i+2},x_i),(x_i,x_{i+1})$, and in M'. In any case, we have 2-subspace determined with

$M'=M\cup L^2(x_i,x_{i+1},x_{i+2})$.

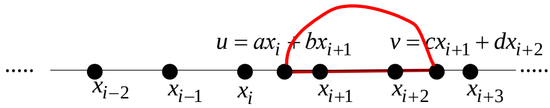
In some form we should consider the role of the vectors x_i and x_{i+2}. The vector x_{i+2} in the case $i=1$ is also a loop element of the 2-subspace generated from any two linearly independent elements from the kernel 2-subspace and the vector x_{i+3}.

In this case it is certain that $x_i=x_1$, then M' begins with kernel 2-subspace, as already described, and continues through the loop x_{i+2} in a branch 2-subspace. If $i=2$, then in M' we have two loops x_i and x_{i+2}, i.e. two loop 2-subspaces, one of them a kernel 2-subspace and the other one is one branch 2-

subspace. If we have $i \geq 3$, then we have two branch 2-subspaces, one of them is a kernel 2-subspace and two loop 2-subspace.

Sub case 8. $u = ax_i + bx_{i+1}$ **,** $v = cx_{i+2} + dx_{i+3}$ where $ab \neq 0$**,** $cd \neq 0$ and $i > 1$**.**

In this situation the vectors u, x_{i+1}, x_{i+2}, v are four linearly independent



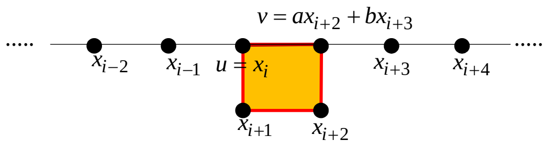
elements. Here, all four pairs of elements $(u, x_{i+1}), (x_{i+1}, x_{i+2}), (x_{i+2}, v), (v, u)$ belong in the new 2-subspace M' and they for themselves form a cyclic 2-subspace, which at the same time is 2-subspace which is also in X^2 and is a part of M'. Now, it is not clear whether we can consider the elements u and v everyone of them separately for loops of two 2-subspces.

Sub case 9. $u = ax_1 + bx_2$, $v = cx_3 + dx_4$, where $ab \neq 0$, $cd \neq 0$

This sub case is fully described in sub case 7 from this case.

Sub case 10. $u = x_i$, $v = ax_{i+2} + bx_{i+3}$**,** $i > 1$

In this situation we have a sub case which is analogous to the previous case. But, now since $i > 1$, we get that the vectors $x_1, x_2, ..., x_{i-1}$ are linearly



independent and the pairs of 2-vectors $(x_1, x_2), (x_2, x_3), ..., (x_{i-2}, x_{i-1})$ form finite branch 2-subspace. Additionally, in relation to the previous sub case, the vector $u = x_i$ will be a loop element of a loop 2-subspace.

Sub case 11. $u = x_i$, $v = \alpha_j x_j + \alpha_{j+1} x_{j+1} + \alpha_{j+2} x_{j+2} + \alpha_{j+3} x_{j+3}$, where i can be any positive integer, as well as j, and $\alpha_j \alpha_{j+3} \neq 0$.

Here, we are interested only the end cases, because the remaining case is the same as for the two-sided branch.

Situation 1. $u = x_1$, $v = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$, where $\alpha_1 \alpha_4 \neq 0$.

The vector v is not a coordinate of any 2-vector from M . Enough reason for this is the fact that $\alpha_1, \alpha_4 \neq 0$. According to this, the branch generated by $x_1, x_2, x_3, ...$ is simply supplemented with a first element $v = x_\circ$, and that will be a new branch $v = x_\circ, x_1, x_2, x_3, ...$.

Situation 2. $u = x_2$, $v = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$, where $\alpha_1 \alpha_4 \neq 0$.

From the condition $\alpha_1, \alpha_4 \neq 0$, we have that the vector v is not a coordinate of any 2-vector from M . According to this, the 2-vectors $(x_1, x_2), (v, x_2), (x_3, x_2)$ form a loop 2-subspace S' which has the form

$$S' = \bigcup_{w \in L(x_1, v, x_3)} L(w, x_2) \times L(w, x_2) .$$

The vector x_4 is ending vector of an one-sided infinite branch, i.e. $x_4, x_5, ...$ We will denote that 2-subspace with S. So,

$M' = S' \cup S$.

Situation 3. $u = x_3$, $v = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$, where $\alpha_1 \alpha_4 \neq 0$,

From the condition $\alpha_1, \alpha_4 \neq 0$, we have that the vector v is not a coordinate of any 2-vector from M. Now, on the vector $u = x_3$, as basic vector, one 2-subspace S' is generated and it is a loop 2-subspace. Its generator elements are $(u, x_2), (u, x_4), (u, v)$. The vector (x_1, x_2), which is attached to itself in this loop 2-subspace, which is trivial. That is the subspace $S" = \{A(x_1, x_2) / A \in M_2(\Phi)\}$. From the other side the vectors $x_4, x_5, x_6, x_7, ...$ form one-sided branch 2-subspace S''' which is also attached to the loop 2-subspace which is previously described. According to this, in this case the extension is

$M' = S' \cup S" \cup S'''$

Situation 4. $u = x_4$, $v = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$, where $\alpha_1 \alpha_4 \neq 0$,

In this situation we have similar position as in the previous three cases. Because of the condition $\alpha_1 \alpha_4 \neq 0$, the vector v can not be a coordinate in any vector from M. Now, the vector $u = x_4$ will become loop centre of the three 2-vectors $(u, x_3), (u, x_5), (u, v)$ which form a loop 2-subspace S'. The previous 2-vectors $(x_1, x_2), (x_2, x_3)$ form finite branch 2-subspace which will be denoted with $S"$, and the vectors from the sequence $x_5, x_6, ...$ form branch 2-subspace which will be denoted with S'''. Finally, this 2-subspace will be

$M' = S' \cup S" \cup S'''$.

Situation 5. $u = x_5$, $v = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$, where $\alpha_1 \alpha_4 \neq 0$,

This situation is almost identical to the previous situation. Here, the starting branch 2-subspace $S"$ is generated by the 2-vectors $(x_1, x_2), (x_2, x_3), (x_3, x_4)$. The vector $u = x_5$ is a vector which is a loop element for the new 2-subspace M', which will be denoted as S'. It is a basis of the 2-vectors $(x_4, u), (x_6, u), (v, u)$. On the other hand, the vectors $x_6, x_7, ...$ form a branch 2-subspace S'''. So,

$M' = S' \cup S" \cup S'''$.

Situation 6. $u = x_1$, $v = \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5$, where $\alpha_2 \alpha_5 \neq 0$,

The proof is trivial. It is the same as if to the vector x_1 is added one vector, and now the vectors $v, x_1, x_2, x_3, ...$ form a branch 2-subspace. Certainly, the vector v does not belong in any coordinate of the 2-vector from M'.

Situation 7. $u = x_1$, $v = \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5 + \alpha_6 x_6$, where $\alpha_3 \alpha_6 \neq 0$,

The proof is trivial. It is the same as if to the vector x_1 is added one vector, and now the vectors $v, x_1, x_2, x_3, ...$ form a branch 2-subspace. Certainly, the vector v does not belong in any coordinate of the 2-vector from M'.

In this situation we have one vector which is a coordinate of a 2-vector from M and one vector which is not a coordinate of a 2-vector from M. The discussion here is same as in all other cases. The rest of the cases. i.e. when we have indexes i and $j, j+1, j+2, j+3$, where $i<j$ we have totally analogous situation as in the situation 7, which is true for every $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}+\alpha_{j+3}x_{j+3}$, $j>3$.

Sub case 11'. $u=x_i$, $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}+\alpha_{j+3}x_{j+3}$, $i>1$ where i can be any positive integer greater than 1, same as j, where $\alpha_j\alpha_{j+3}\neq 0$.

We should mention here that one vector u is always a coordinate of 2-vector from M and the other vector v is not a coordinate of none of the 2-vectors from M .

Here, we are interested only the ending cases, because the rest of the situations are same as the ones for the two-sided branch.

Situation 1. $u=x_i$, $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$, where $\alpha_{i-1}\alpha_{i+2}\neq 0$

Situation 2. $u=x_i$, $v=\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}+\alpha_{i+3}x_{i+3}$, where $\alpha_i\alpha_{i+3}\neq 0$

Situation 3. $u=x_i$, $v=\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}+\alpha_{i+3}x_{i+3}+\alpha_{i+4}x_{i+4}$, where $\alpha_{i+1}\alpha_{i+4}\neq 0$,

Situation 4. $u=x_i$, $v=\alpha_{i+2}x_{i+2}+\alpha_{i+3}x_{i+3}+\alpha_{i+4}x_{i+4}+\alpha_{i+5}x_{i+5}$, where $\alpha_{i+2}\alpha_{i+5}\neq 0$,

Situation 5. $u=x_{i+1}$, $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$, where $\alpha_{i-1}\alpha_{i+2}\neq 0$,

Situation 6. $u=x_{i+2}$, $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$, where $\alpha_{i-1}\alpha_{i+2}\neq 0$,

Situation 7. $u=x_{i+3}$, $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$, where $\alpha_{i-1}\alpha_{i+2}\neq 0$,

Situation 8. $u=x_{i+4}$, $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$, where $\alpha_{i-1}\alpha_{i+2}\neq 0$,

The discussion are as in all the other cases. The rest of the cases, i.e. when we have indexes i and $j, j+1, j+2, j+3$, where $i<j$ we have totally analogous situation as in situation 4, which is true for every $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}+\alpha_{j+3}x_{j+3}$, $j>i$

Sub case 12. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}+\alpha_{j+3}x_{j+3}$, $i>1$ where $\alpha_j\alpha_{j+3}\neq 0$.

This case is possible because the element v is not a coordinate of none of the elements of the 2-subspace M , but it is still element of the vector space X . Same case can be considered also for a vector v, in the form $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}+\alpha_{j+3}x_{j+3}+...+\alpha_{j+\kappa}x_{j+\kappa}$, for any k which is bigger than 3, but here $\alpha_j\alpha_{j+k}\neq 0$. There is no essential difference from this aspect.

In all this situations, for the vector v, we have $\alpha_{i-1}\alpha_{i+2}\neq 0$, $\alpha_i\alpha_{i+3}\neq 0$, $\alpha_{i+1}\alpha_{i+4}\neq 0$, $\alpha_{i+2}\alpha_{i+5}\neq 0$ and $\alpha_{i-1}\alpha_{i+2}\neq 0$. With this it is generally obtained that this vector in neither of this situations can not belong. i.e. to be a coordinate of any 2-vector from M .

Situation 1. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$, where $\alpha_{i-1}\alpha_{i+2}\neq 0$.

The vector $u=ax_i+bx_{i+1}$,belongs to the 2-subspace $S=\left\{A(x_i,x_{i+1})/A=\begin{bmatrix}\alpha_{i=1} & 0\\ 0 & \alpha_i\end{bmatrix}\right\}$

Since v doesn't belong in M , this situation is like the sub case 2 of case 2.

Situation 2. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}+\alpha_{i+3}x_{i+3}$, where $\alpha_i\alpha_{i+3}\neq 0$.

The vector $u = ax_i + bx_{i+1}$, belongs to the 2-subspace $S = \left\{ A(x_i, x_{i+1}) / A = \begin{bmatrix} \alpha_i & 0 \\ 0 & \alpha_{i+1} \end{bmatrix} \right\}$. Since v doesn't belong in M, this situation is like the sub case 2 of case 2.

Situation 3. $u = \alpha_i x_i + \alpha_{i+1} x_{i+1}$, $v = \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2} + \alpha_{i+3} x_{i+3} + \alpha_{i+4} x_{i+4}$, where $\alpha_{i+1}\alpha_{i+4} \neq 0$

The vector $u = ax_i + bx_{i+1}$, belongs to the 2-subspace $S = \left\{ A(x_i, x_{i+1}) / A = \begin{bmatrix} \alpha_i & 0 \\ 0 & \alpha_{i+1} \end{bmatrix} \right\}$. Since v doesn't belong in M, this situation is like the sub case 2 of case 2.

Situation 4. $u = \alpha_i x_i + \alpha_{i+1} x_{i+1}$, $v = \alpha_{i+2} x_{i+2} + \alpha_{i+3} x_{i+3} + \alpha_{i+4} x_{i+4} + \alpha_{i+5} x_{i+5}$, where $\alpha_{i+2}\alpha_{i+5} \neq 0$.

Same as in situation 3.

Situation 5. $u = \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$, $v = \alpha_{i-1} x_{i-1} + \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$, where $\alpha_{i-1}\alpha_{i+2} \neq 0$.

Same as in situation 3. Here, for the vector $u = \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$ belongs in the 2-subspace $S = \left\{ A(x_{i+1}, x_{i+2}) / A = \begin{bmatrix} \alpha_{i+1} & 0 \\ 0 & \alpha_{i+2} \end{bmatrix} \right\}$.

Situation 6. $u = \alpha_{i+2} x_{i+2} + \alpha_{i+3} x_{i+3}$, $v = \alpha_{i-1} x_{i-1} + \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$, where $\alpha_{i-1}\alpha_{i+2} \neq 0$.

Same as in situation 3. Here, for the vector $u = \alpha_{i+2} x_{i+2} + \alpha_{i+3} x_{i+3}$ belongs in the 2-subspace $S = \left\{ A(x_{i+2}, x_{i+3}) / A = \begin{bmatrix} \alpha_{i+2} & 0 \\ 0 & \alpha_{i+3} \end{bmatrix} \right\}$.

Situation 7. $u = \alpha_{i+3} x_{i+3} + \alpha_{i+4} x_{i+4}$, $v = \alpha_{i-1} x_{i-1} + \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$, where $\alpha_{i-1}\alpha_{i+2} \neq 0$.

Same as in situation 3. Here, for the vector $u = \alpha_{i+3} x_{i+3} + \alpha_{i+4} x_{i+4}$ belongs in the 2-subspace $S = \left\{ A(x_{i+3}, x_{i+4}) / A = \begin{bmatrix} \alpha_{i+3} & 0 \\ 0 & \alpha_{i+4} \end{bmatrix} \right\}$.

Sub case 12'. $u = \alpha_1 x_1 + \alpha_2 x_2$, $v = \alpha_j x_j + \alpha_{j+1} x_{j+1} + \alpha_{j+2} x_{j+2} + \alpha_{j+3} x_{j+3}$, where $\alpha_j \alpha_{j+3} \neq 0$.

In all next situations of this sub case we have that the vector belongs in the 2-subspace generated from one element, i.e. from the element $u = \alpha_i x_i + \alpha_{i+1} x_{i+1}$. The vector $v = \alpha_j x_j + \alpha_{j+1} x_{j+1} + \alpha_{j+2} x_{j+2} + \alpha_{j+3} x_{j+3}$ is not a coordinate of none of the vectors in the 2-subspace M'. Here, the vectors x_1, v, x_2 are linearly independent. In the beginning of the branch they form a loop 2-subspace. The sequence of vectors $x_3, x_4, x_5, x_6, ...$ form a branch 2-subspace. According to this, for any of the next four possible situations, we have that the 2vector subspace has form $M' = S' \cup S''$,

where $S' = \bigcup_{w \in L(x_1, v, x_2)} L(w,u) \times L(w,u)$, a S'' e $S'' = \bigcup_{k=4}^{+\infty} \bigcup_{\alpha_{i-1}, \alpha_{i+1}} L^2(\alpha_{i-1} x_{i-1} + \alpha_{i+1} x_{i+1}, x_2)$.

Situation 1. $u = \alpha_1 x_1 + \alpha_2 x_2$, $v = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4$, where $\beta_1 \beta_4 \neq 0$

Situation 2. $u = \alpha_1 x_1 + \alpha_2 x_2$, $v = \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5$, where $\beta_2 \beta_5 \neq 0$

Situation 3. $u = \alpha_1 x_1 + \alpha_2 x_2$, $v = \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_6 x_6$, where $\beta_3 \beta_6 \neq 0$

Situation 4. $u=\alpha_1x_1+\alpha_2x_2$, $v=\beta_4x_4+\beta_5x_5+\beta_6x_6+\beta_7x_7$, where $\beta_4\beta_7\neq 0$

In all situations from situation 2 to situation 4 we have exactly the same position: one vector that belongs to the starting branch 2-subspace. The second vector is not a coordinate of none of the 2-vectors that belong in the starting 2-subspace. Everywhere we have a situation in which the sub case from case 2 is repeating.

Sub case 13. $u=\alpha_ix_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}+\alpha_{i+3}x_{i+3}+...+\alpha_{i+k}x_{i+k}$ and $v=\alpha_jx_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}+\alpha_{j+3}x_{j+3}+...+\alpha_{j+s}x_{j+s}$, where $k,s\geq 3$. Here i and j can be any positive integers.

In this situation, nether the vector u nor the vector v are not coordinates of a 2-vector from M , so, according to this, this sub case is the same as the case 1.

Sub case 14. $u=\alpha_1x_1+\alpha_2x_2$, $v=\beta_jx_j+\beta_{j+1}x_{j+1}+\beta_{j+2}x_{j+2}$, where $\alpha_i\alpha_{i+1}\neq 0$ and $\beta_j\beta_{j+2}\neq 0$

This sub case is absolutely possible, and u and v are vectors which are coordinates of some 2-vectors from the 2-subspace M . Now we have

Situation 1. $v=\beta_1x_1+\beta_2x_2+\beta_3x_3$, $\beta_1\beta_2\neq 0$

Certainly, we have here elements which are parts from the 2-subspace M . One such element is certainly the element (x_2,v). Now, because $(x_2,u),(u,v)$ are also from M', we get that the kernel 2-subspace $L^2(u,v,x_2)$ is a 2-subspace from M'. Now, additionally we have that the vector u is a loop for x_1,x_2,v which is a loop 2-subspace, and will be denoted with S. The vectors $x_3,x_4,....$ form a one-sided branch 2-subspace which will be denoted with S'. So,

$M'=M\cup L^2(x_1,u,v)\cup S\cup S'$.

Situation 2. $v=\beta_2x_2+\beta_3x_3+\beta_4x_4$, $\beta_2\beta_4\neq 0$.

It is clear that the vectors x_1,x_2,v form a loop 2-subspace around the loop vector u, which will be denoted with S. On the other hand, the 2-vectors $(u,x_2),(x_2,x_3),(x_3,v),(v,u)$ which belong in M', form a cyclic 2-subspace which will be denoted with S'. The vectors $x_5,x_6,....,x_n,...$ also form one-sided branch 2-subspace which will be denoted with S''. So, we have that

$M'=M\cup S\cup S'\cup S''$

Situation 3. $v=\beta_3x_3+\beta_4x_4+\beta_5x_5$, $\beta_3\beta_5\neq 0$,

In this situation we have totally analogous case to the previous situation, but here the cyclic 2-subspace now is generated by five vectors i.e. $(u,x_2),(x_2,x_3),(x_3,x_4),(x_4,v),(v,u)$. The rest of the elements from the structure are completely the same as in the previous situation. So,

$M'=M\cup S\cup S'\cup S''$

Sub case 14' $u=\alpha_ix_i+\alpha_{i+1}x_{i+1}$, $v=\beta_jx_j+\beta_{j+1}x_{j+1}+\beta_{j+2}x_{j+2}$, where $\alpha_i\alpha_{i+1}\neq 0$ and $\alpha_j\alpha_{j+2}\neq 0$, and $i>1$.

This case is absolutely possible, where u and v are vectors which are coordinates in some 2-vectors from the 2-subspace M .

Situation 1. $u=\alpha_ix_i+\alpha_{i+1}x_{i+1}$, $v=\beta_{i-1}x_{i-1}+\beta_ix_i+\beta_{i+1}x_{i+1}$

From the definition of u and v is clear that the 2-vectors $(x_i,u),(u,v),(v,x_i)$ form a kernel 2-subspace from M', which will be denoted with S. The vectors $x_1,x_2,...,x_{i-2}$ (if there's any, i.e. if $i>3$) form a finite branch 2-subspace which will be denoted with S'. The vector x_{i+1} will be considered for a kernel 2-subspace which in a loop connects any two vectors from the kernel of the extension, together with the vector x_{i+2}. This 2-subspace from M' will be denoted with S''. At the end, the vectors $x_{i+3},x_{i+4},...$ form one sided branch 2-subspace which will be denoted with S'''.

So, this extension is $M'=M\cup S'\cup S''\cup S'''\cup S$.

Situation 2. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\beta_i x_i+\beta_{i+1}x_{i+1}+\beta_{i+2}x_{i+2}$.

This situation is completely analogous to the previous situation 1.

Situation 3. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\beta_{i+1}x_{i+1}+\beta_{i+2}x_{i+2}+\beta_{i+3}x_{i+3}$.

This situation is analogous to the situation 2 from the previous sub case. The discussion is the same.

Situation 4. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\beta_{i+2}x_{i+2}+\beta_{i+3}x_{i+3}+\beta_{i+4}x_{i+4}$.

This situation is analogous to the situation 3 from the previous sub case. The discussion is the same.

Sub case 15. $u=x_i$, $i>1$, $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}$, where $\alpha_j\alpha_{j+2}\neq 0$

This case is absolutely possible, and here u and v are vectors which are coordinates of some 2-vectors from the 2-subspace M .

Situation 1. $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $\alpha_{i-1}\alpha_{i+1}\neq 0$,

In this situation is enough to suppose that $u=x_2$. The rest of the situations are considered as an addition to this situation. If $u=x_2$, then from $v=\alpha_1x_1+\alpha_2x_2+\alpha_3x_3$, we get that essentially we do not have an extension of M , because $(u,v)=(x_2,\alpha_1x_1+\alpha_2x_2+\alpha_3x_3)\in M$. So, in this situation

$M'=M$

In the situation when $i>2$, the vector x_{i-1} is a loop element of the 2-subspace in which are included as 2-vectors $(x_{i-2},x_{i-1}),(v,x_{i-1}),(x_{i+1},x_{i-1})$. That 2-subspace will be denoted with S . Additionally one more 2-subspace can appear, and that is a finite branch 2-subspace, which will be generated from the vectors $x_1,x_2,....,x_{i-2}$, and will be denoted with S'. So, now we have that

$M'=M\cup S\cup S'$.

Situation 2. $v=\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$, $\alpha_i\alpha_{i+2}\neq 0$.

In this situation we will make additional assumptions that $u=x_2$. In this situation we have that $v=\alpha_2x_2+\alpha_3x_3+\alpha_4x_4$, where $(x_3,v),(x_3,x_2)$ are vectors from M , and (v,u) is a 2-vector which we add. So, we have a real extension, where those three vectors form a kernel 2-subspace from X^2, denoted with S . Additionally, the vector u is a loop for the 2-subspace from M', in which the 2-vector (x_1,x_2) is included also, and is denoted with S'. In this 2-subspace there will be another loop element, and that is the vector x_3, which will be denoted with S'', and in it belong any two vectors from the kernel 2-subspace, together with the vector $(x_1, x_2) = (x_1, u)$. So, in this case we have that

$M' = M \cup S \cup S' \cup S''$.

We have a completely analogous situation when the vector $u = x_2$ will be any vector $u = x_i, i > 2$, and here additionally we will have one more branch 2-subspace S'''. In this case we would have that

$M' = M \cup S \cup S' \cup S'' \cup S'''$.

Situation 3. $v = \alpha_{i+1}x_{i+1} + \alpha_{i+2}x_{i+2} + \alpha_{i+3}x_{i+3}$, $\alpha_{i+1}\alpha_{i+3} \neq 0$.

In this situation we well make the same additional assumptions, for which $u = x_2$, and the case $u = x_i$ will be considered additionally. In this situation we have that the vector $u = x_2$ is a loop 2-vector for the 2-vectors $(u, x_1), (u, x_3), (u, v)$ and the 2-subspace that they generate will be denoted with S. The vectors $(u, x_3), (u, x_4), (x_4, v), (v, u)$ are four vectors which form a cyclic 2-subspace, denoted with S'. Additionally, the 2-vectors $x_5, x_6, ..., x_n, ...$ form one sided branch 2-subspace, denoted with S''. Finally, this extension of M will be

$M' = M \cup S \cup S' \cup S''$.

If we have extension with a vector $u = x_i$, $i > 2$ then additionally we will have one more 2-subspace which is a finite branch generated with the vectors $x_1, x_2, ..., x_{i-1}$ and which will be denoted with S'''. In that case we will have an extension in the following form

$M' = M \cup S \cup S' \cup S'' \cup S'''$.

Situation 4. $v = \alpha_{i+2}x_{i+2} + \alpha_{i+3}x_{i+3} + \alpha_{i+4}x_{i+4}$, $\alpha_{i+2}\alpha_{i+4} \neq 0$.

In this situation we have totally analogous picture as in the previous example, except that in this case we will have that the number of generator elements of the cyclic 2-subspace is for one 2-vector bigger than before.

Sub case 15'. $u = x_1$ $v = \alpha_j x_j + \alpha_{j+1}x_{j+1} + \alpha_{j+2}x_{j+2}$, where $\alpha_j \alpha_{j+2} \neq 0$

This case is absolutely possible, and here u and v are vectors which are coordinates of some 2-vectors from the 2-subspace M . Here, we should especially see the case when $u = x_1$, and $j \geq 1$ is any positive integer. The previous cases are completely analogous as in the example of two-sided branch.

Situation 1. $v = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, $\alpha_1 \alpha_3 \neq 0$.

In this situation we have that the 2-vectors $(u, x_2), (x_2, v), (v, u)$ which belong in M form also a kernel 2-subspace S in M'. According to this, in this situation we have a kernel 2-subspace, which is connected to a loop 2-subspace S' and at the end of the last kernel 2-subspace is attached a one sided 2-subspace. So,

$M' = M \cup S \cup S'$.

Situation 2. $v = \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$, $\alpha_2 \alpha_4 \neq 0$

In this situation we have one cyclic 2-subspace determined with the 2-vectors

$(u, x_2), (x_2, x_3), (x_3, v), (v, u)$, which will be denoted as S. On the other hand the 2-vectors $(x_4,x_3),(x_4,x_5),(x_4,v)$ form a loop 2-subspace which will be denoted as S'. The rest of the vectors $x_5,x_6,x_7,x_8,...,x_n,...$, are certainly a part of M. According to this,

$M'=M\cup S\cup S'$

Situation 3. $v=\alpha_3x_3+\alpha_4x_4+\alpha_5x_5$, $\alpha_3\alpha_5\neq 0$.

In this situation we have only a movement for one element in comparison with the previous case. Here, the generator elements of the cyclic 2-subspace are the elements $(u,x_2),(x_2,x_3),(x_3,x_4),(x_4,v),(v,u)$, and this cyclic 2-subspase we will denote as S. The loop 2-subspace we will denote with S', and it is generated with the vectors $(x_5,x_4),(x_5,x_6),(x_5,v)$. The rest of the vectors $x_6,x_7,x_8,...,x_n,...$ form a one sided branch 2-subspace. So,

$M'=M\cup S\cup S'$.

Situation 4. $v=\alpha_4x_4+\alpha_5x_5+\alpha_6x_6$, $\alpha_4\alpha_6\neq 0$.

This case is the same as the previous case, just moved for one element to the right.

Sub case 16. $v=\alpha_jx_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}$, $u=\beta_ix_i+\beta_{i+1}x_{i+1}+\beta_{i+2}x_{i+2}$, $i<j$, where $\alpha_j\alpha_{j+2}\neq 0$ and $\beta_i\beta_{i+2}\neq 0$.

It is clear that for any i and j we have that the smallest variant for u and v is that they have the following form $u=\alpha_jx_j+\alpha_{j+2}x_{j+2}$ and $v=\beta_ix_i+\beta_{i+2}x_{i+2}$.

Situation 1. $u=\beta_1x_1+\beta_2x_2+\beta_3x_3$, $v=\alpha_1x_1+\alpha_2x_2+\alpha_3x_3$.

The 2-vectors $(x_2,u),(x_2,v),(u,v)$ belong in the 2-subspace M', so, according to this also the kernel 2-subspace S generated by them is a subset of M'. Additionally the vectors $x_4,x_5,....$ determine a one sided branch 2-subspace, which is a subspace both from M' and from M. It will be denoted with S'. According to this,

$M'=M\cup S\cup S'$

Situation 2. $u=\beta_1x_1+\beta_2x_2+\beta_3x_3$, $v=\alpha_2x_2+\alpha_3x_3+\alpha_4x_4$.

The 2-vectors $(u,x_2),(x_2,x_3),(x_3,u),(u,v)$ are four vectors which form a cyclic 2-subspace from X^2, and because they belong in M', they are a 2-subspace from M', too, which we will denote as S. If we have in mind now also the 2-subspace M, then we get that

$M'=M\cup S$.

Situation 3. $u=\beta_1x_1+\beta_2x_2+\beta_3x_3$, $v=\alpha_3x_3+\alpha_4x_4+\alpha_5x_5$.

We will consider the 2-vectors $(u,x_2),(x_2,x_3),(x_3,x_4),(x_4,v),(v,u)$ which for itself form a cyclic 2-subspace, with five generator elements, which we will denote as S. With necessary calculations made, we get that the 2-subspace M' is in fact determined with

$M'=M\cup S$.

Situation 4. $u=\beta_1x_1+\beta_2x_2+\beta_3x_3$, $v=\alpha_4x_4+\alpha_5x_5+\alpha_6x_6$

In this situation we have a 2-subspace which is equal to the 2-subspace determined in the previous situation 3.

Completely analogous are considered all 2-subspaces which are determined with such 2-vectors. The most that can happen is to have at the beginning a finite branch 2-subspace generated by $x_1, x_2, ..., x_{i-1}$.

Sub case 17. $u = \alpha_i x_i + \alpha_{i+1} x_{i+1}$, $v = \beta_j x_j + \beta_{j+1} x_{j+1}$, $i \le j$, $\alpha_i \alpha_{i+1} \ne 0$, ${}_j\alpha_{j+1} \ne 0$.

Situation 1. $u = \alpha_1 x_1 + \alpha_2 x_2$, $v = \beta_1 x_1 + \beta_2 x_2$

In this situation we have a 2-vector which belong to the 2-subspace M', and that is the 2-vector (u, v), for which all is already defined, like the value $\Lambda(u, v)$. This is practically the same as the sub case 2 of case 3. In other words, there is no extension of M.

Situation 2. $u = \alpha_1 x_1 + \alpha_2 x_2$, $v = \beta_2 x_2 + \beta_3 x_3$,

In this situation, the two vectors belong in the 2-subspace M, but here we have an extension of M. Regarding the extension, we have a completely same situation as in the sub case 7. So, the vectors x_1, x_2, x_3 essentially form a kernel 2-subspace, and to it a one sided branch 2-subspace is attached. In this situation we would have that

$$M' = M \cup L(x_1, x_2, x_3) \times L(x_1, x_2, x_3).$$

Situation 3. $u = \alpha_1 x_1 + \alpha_2 x_2$, $v = \beta_3 x_3 + \beta_4 x_4$

In this we have that the four 2-vectors $(u, x_2), (x_2, x_3), (x_3, v), (v, u)$ form cyclic 2-subspace, denoted as S. Now, the extension of M would be

$$M' = M \cup \bigcup_{w \in L(v, x_1, x_2)} L(w, u) \times L(w, u) \cup \bigcup_{z \in L(u, x_3, x_4)} L(z, v) \times L(z, v) \cup S$$

Situation 4. $u = \alpha_1 x_1 + \alpha_2 x_2$, $v = \beta_j x_j + \beta_{j+1} x_{j+1}$, $j > 3$.

This situation is the same as the previous situation, but here the number of the 2-vectors that form the cyclic 2-subspace is bigger. Here, those 2-vectors that form the cyclic 2-subspace are $(u, x_2), (x_2, x_3), (x_3, x_4), ..., (x_j, v)(v, u)$ and as before, it will be denoted with S. In this situation we would have an extension in the form

$$M' = M \cup \bigcup_{w \in L(v, x_1, x_2)} L(w, u) \times L(w, u) \cup \bigcup_{z \in L(u, x_j, x_{j+1})} L(z, v) \times L(z, v) \cup S$$

Situation 5. $u = \alpha_i x_i + \alpha_{i+1} x_{i+1}$, $v = \beta_i x_i + \beta_{i+1} x_{i+1}$, $i > 1$.

In this situation, we don't have extension of the 2-subspace M, because (v, u) is a vector from M, which belongs to the 2-subspace

$$S = \{(x, y) / A(x_i, x_{i+1}), A \in M_2(\Phi)\} \subseteq M.$$

Situation 6. $u = \alpha_i x_i + \alpha_{i+1} x_{i+1}$, $v = \beta_{i+1} x_{i+1} + \beta_{i+2} x_{i+2}$, $i > 1$.

In this situation we have a real extension of the 2-subspace M. Now, the kernel 2-subspace generated by the 2-vectors $(u, x_{i+1}), (x_{i+1}, v), (v, u)$ in fact, same as in sub case 7 from this case, is extended to 2-subspace which is determined with $(x_i, x_{i+1}), (x_{i+1}, x_{i+2}), (x_{i+2}, x_i)$ and it will be denoted with S. So, we have

$$M' = M \cup \bigcup_{w \in L(v, x_i, x_{i+1})} L(w, u) \times L(w, u) \cup \bigcup_{z \in L(u, x_{i+1}, x_{i+2})} L(z, v) \times L(z, v) \cup S$$

Situation 7. $u = \alpha_i x_i + \alpha_{i+1} x_{i+1}$, $v = \beta_{i+2} x_{i+2} + \beta_{i+3} x_{i+3}$,, $i > 1$.

In this situation we have an extension of the 2-subspace M. Now, the 2-vectors $(u,x_{i+1}),(x_{i+1},x_{i+2}),(x_{i+2},v),(v,u)$ form a cyclic 2-subspace. On the other hand, the vectors $x_{i+3},x_{i+4},...,x_n,...$ form a one-sided branch 2-subspace which will be denoted with S, and the vectors u and v are loop centres of the 2-subspaces $\bigcup_{w\in L(v,x_i,x_{i+1})} L(w,u)\times L(w,u)$ and $\bigcup_{z\in L(u,x_{i+2},x_{i+3})} L(z,v)\times L(z,v)$. If $i>2$, then we have also a starting 2-subspace, which is a branch 2-subspace and it will be denoted with S'. So,

$$M'=M\cup \bigcup_{w\in L(v,x_i,x_{i+1})} L(w,u)\times L(w,u)\cup \bigcup_{z\in L(u,x_{i+2},x_{i+3})} L(z,v)\times L(z,v)\cup S\cup S'$$

Situation 8.// $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\beta_j x_j+\beta_{j+1}x_{j+1}$,, $i>1$, $j\ge i+3$.

This situation is completely analogous of the previous situation, and here only is enlarged the cyclic 2-subspace, where its generators are the 2-vectors determined with $(u,x_{i+1}),(x_{i+1},x_{i+2}),(x_{i+2},x_{i+3}),....(x_j,v),(v,u)$. Now, we have

$$M'=M\cup \bigcup_{w\in L(v,x_i,x_{i+1})} L(w,u)\times L(w,u)\cup \bigcup_{z\in L(u,x_j,x_{j+1})} L(z,v)\times L(z,v)\cup S\cup S'.$$

3. EXTENSION OF A 2-SKEW-SYMMETRIC LINEAR FORM

Theorem. *Let $\Lambda : M\to\mathbb{R}$ be a 2-skew-symmetric form such that $\Lambda(x,y)\le p(x,y)$ for every $(x,y)\in M$, $p: X^2\to\mathbb{R}$ be a 2-semi norm and M is a branch 2-subspace of the 2-space X^2. Let M' be an extension of M as in sub case 1 of case 2. Then there exists a 2-skew-symmetric linear form $\Lambda' : M'\to\mathbb{R}$ such that*

$\Lambda'/M=\Lambda$

$-p(-x,y)\le\Lambda(x,y)\le p(x,y)$. (*)

Proof. We will consider the two cases separately.

Situation 1. Let $u=x_i$, where $i>1$. In this situation we have a complete analogy with the one in the case of two sided branch 2-subspace. That is why it is enough to take it from there. [9]

Situation 2. Let $u=x_1$. The choice of two 2-vectors from the 2-subspace of u can be done in a way that it is done in the 2-subspaces generated in the paper for two sided branch 2-subspace. In other words, we can choose 2-vectors in the form $(u,\alpha_1x_1+\alpha_2x_2)$ and $(u,\alpha_1^{'}x_1+\alpha_2^{'}x_2)$. Further on, the proof is the same as the proofs in the theorems of Hahn-Banach in the part for a two sided branch 2-subspace. [9]

The rest of the cases are the same with the corresponding ones described in the paper for two sided branch 2-subspace. [9]

CONFLICT OF INTEREST

No conflict of interest was declared from the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

References

[1] R.Malčeski, A.Malčeski, n-*seminormed space*, Annuaire de l'Institute des Mathématiques, Faculté des Sciences de l'Université "Sv. Kiril et Metodij" - Skopje), 38(1997)

[2] A.Misiak: n-*inner product spaces*, Math.Nachr. 140 (1989)

[3] S.Gähler, *Lineare 2-normierte Raume,* Math.Nach. 28(1965)

[4] A.Malčeski, *Zabeleška za definicijata na 2-normiran prostor,* Mat. Bilten, Tom 26 (2002)

[5]A,Malčeski, V. Manova Erakovik, *Algebraic structure of the kernel of the n-seminorm*, Matematički bilten, 31, (LVII), Makedonija, (2007)

[6] A.Malčeski, V. Manova Erakovikj. *An extended type of Hahn-Banach for Skew-Symmetric linear forms.* Mat. Bilten, 35(LXI)Tome, 2011, pp 41-49

[7] A.Malčeski, V.Manova Erakovik, *Some 2-subspaces of 2-space*, Математички Билтен, 35(LXI), Makedonija, (2007)

[8] D.Mitrinović, *Polinomi i Matrici*, Naučna Knjiga, Beograd (1991)

[9] S.Brsakoska, A.Malčeski, *Extension of two sided branch 2-subspace and some extensions of Hahn-Banach type for skew-symmetric 2-linear functionals defined on it,* Proceedings od CODEMA 2020

[1)]Faculty for Natural Sciences and Mathematics, University "Sts. Cyril and Methodius", Skopje, Republic of N. Macedonia
E-mail address: sbrsakoska@gmail.com
[2)]Faculty Mechanical Engineering, University "Sts. Cyril and Methodius", Skopje, Republic of N. Macedonia
E-mail address: aleksa.malceski@gmail.com

Publisher
Union od Mathematicians of Macedonia – ARMAGANKA

Editor in chief
Prof. d-r Aleksa Malcheski

CIP - Каталогизација во публикација
Национална и универзитетска библиотека "Св. Климент Охридски", Скопје

51(082)

PROCEEDINGS of CODEMA 2020 = Зборник на трудови од CODEMA 2020. -
Skopje : Armaganka, 2021. - 375 стр. : илустр. ; 25 см

Текст на мак. и англ. јазик. - Фусноти кон текстот. - Библиографија кон трудовите

ISBN 978-608-4904-09-0

а) Математика -- Зборници

COBISS.MK-ID 53570309