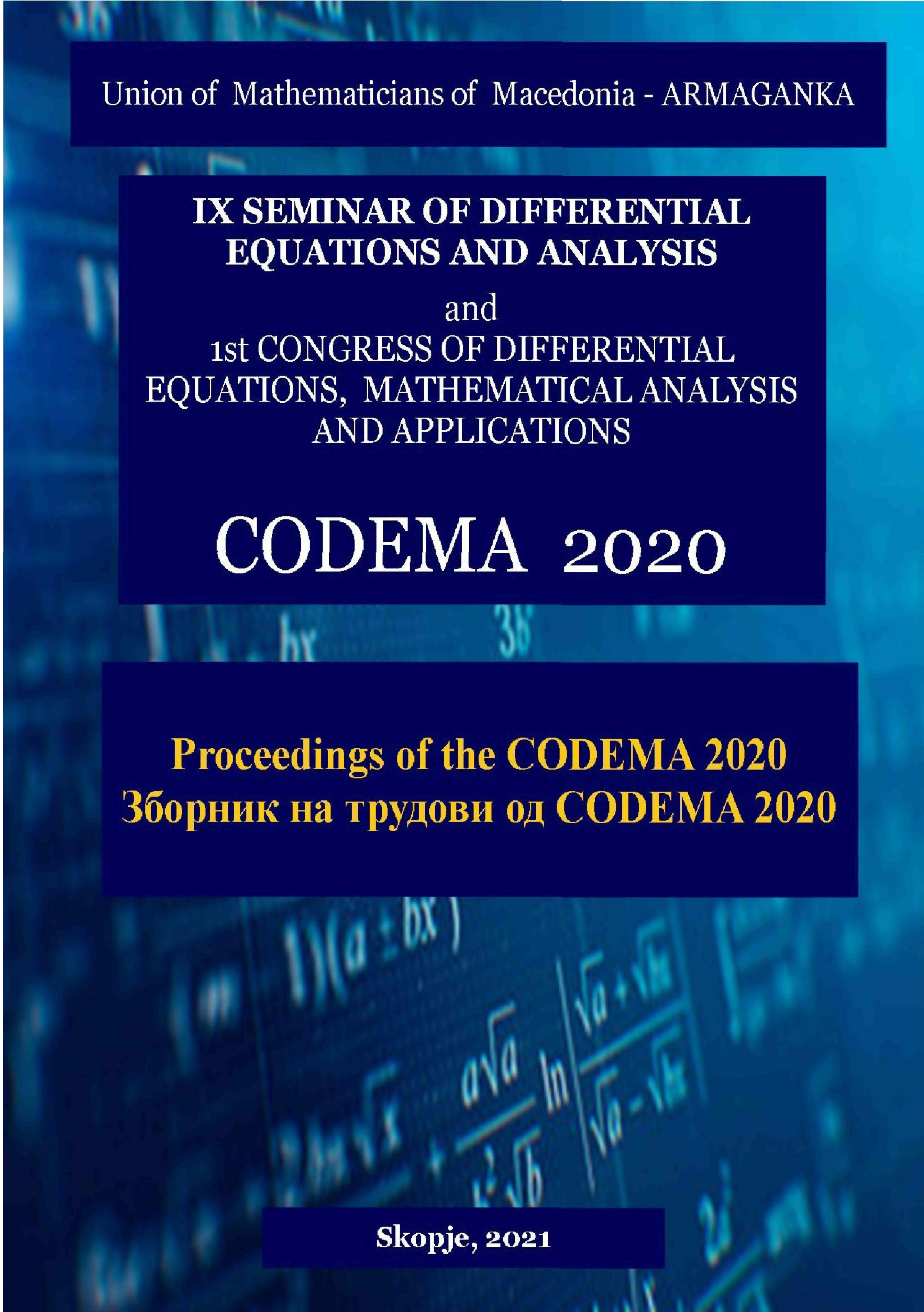

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EXTENSION OF TWO SIDED BRANCH 2-SUBSPACE AND SOME EXTENSIONS OF HAHN - BANACH TYPE FOR SKEW-SYMMETRIC 2-LINEAR FUNCTIONALS DEFINED ON IT

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Abstract. In this paper 2-subspaces from 2-space X^2, which are from two sided branch 2-subspace type, will be taken in consideration. Then all its possible extensions adding one element (u,v) and their complete description will be considered. Also, all extensions of 2-skew-symmetric linear form defined on 2-subspace M' Hahn-Banach type will be considered, in the cases when one vector belongs in 2-vector from M, and the other does not belong (u belongs and v does not belong and vice versa), as well as cases when the two coordinates (u,v) do not belong in M.

1. INTRODUCTION

Extensions of mappings is something that is often looked at in various mathematical disciplines. One classical example of extension of a given mapping is of course the Hanh-Banach theorem for linear functional. One version of it comprises the contents of the following theorem.

Theorem 1. *Let* M *be a vector subspace of the vector space* X. *The functional* $p: X \to \mathbb{R}$ *satisfies the conditions*

a) $p(x+y) \le p(x) + p(y)$

b) $p(tx) = tp(x)$,

for every $x, y \in X$ *and* $t \ge 0$.

The functional $f: M \to R$ is linear and $f(x) \le p(x)$. There exists a linear functional $\Lambda: X \to \mathbb{R}$ such that $\Lambda / M = f$ and $-p(-x) \le \Lambda(x) \le p(x)$.

From the title of this paper and the indicated Hahn-Banach theorem it is clear that we need at least the definitions of 2-seminorm and skew-symmetric 2-form. But in order to have the whole picture, we will define 2-norm as well.

Definition 0. Let X be a vector space over the field Φ. The mapping $\| \bullet, \bullet \|: X^2 \to \mathbb{R}_{\ge 0}$ for which the following conditions are fulfilled

(i) $\| x, y \| = 0$ if and only if $\{x, y\}$ is a linear dependent set

(ii) $\| x, y \| = \| y, x \|$ for any $x, y \in X$

(iii) $\| \alpha x, y \| = |\alpha| \cdot \| x, y \|$ for any $\alpha \in \Phi$ and any $x, y \in X$

(iv) $\| x + x', y \| \le \| x, y \| + \| x', y \|$, for any $x, y \in X$,

we call **2-norm**, and $(X^2, \| \bullet, \bullet \|)$ we call **2-normed space**.

Definition 1. Let X be a vector space over the field Φ. The mapping $p: X^2 \to \mathbb{R}_{\ge 0}$ for which the following conditions are fulfilled

(i) $p(x, y) \ge 0$ if $\{x, y\}$ is a linear dependent set

(ii) $p(x,y)=p(y,x)$ for any $x,y\in X$

(iii) $p(\alpha x,y)=|\alpha|\cdot p(x,y)$ for any $\alpha\in\Phi$ and any $x,y\in X$

(iv) $p(x+x',y)\le p(x,y)+p(x',y)$, for any $x,y\in X$,

we call **2-seminorm**, and (X^2,p) we call **2-seminormed space**.

It is worth mentioning that for any 2-norm, it is fulfilled the equation $\|x,y\|=\|x,y+\alpha x\|$, for any $x,y\in X$ and any scalar $\alpha\in\Phi$.

Due to the definition of an n-norm and the definition of an n-semi norm it turned out that, on the set X^2, where X is a vector space over the field Φ (Φ is the field of real numbers or the field of complex numbers), it is convenient to consider additional operations, two of which are partial and one of which is a complete operation, with the aim of making the notation and considerations easier.

One of the corollaries of the last inequality, is a part of every definition of 2-norm, as well as of 2-seminorm, the definition of skew-symmetric 2-form, is given with the following definition of operations in X^2.

Definition 1'. Let X be a vector space over the field Φ. The set X^2 together with the operations

$$(x,z)+(y,z)=(x+y,z)$$

$$(z,x)+(z,y)=(z,x+y)$$

$$A(x,y)=A(x,y)^T$$

where $x,y,z\in X$ and $A\in M_2(\Phi)$ is called a **2-vector space** or **2-space**.

Comment. The third operation in the previous definition is a complete operation, and on the right-hand side of the equality is a multiplication of a matrix with a vector.

Definition 2. Let X be a vector space over the field Φ. The functional $\Lambda:X^2\to\Phi$ for which the following conditions hold

(a) $\Lambda(x+y,z)=\Lambda(x,z)+\Lambda(y,z)$, for arbitrary $x,y,z\in X$

(b) $\Lambda(x,y)=-\Lambda(y,x)$, for arbitrary $x,y\in X$

(c) $\Lambda(\alpha x,y)=\alpha\Lambda(x,y)$, for arbitrary $x,y\in X$ and $\alpha\in\Phi$,

is called **2-skew-symmetric linear form**.

It is not hard to prove that the previous definition (Definition 2) is equivalent with the following definition.

Definition 3. Let X be a vector space over the field Φ. The functional $\Lambda:X^2\to\Phi$ for which the following conditions hold

(a) $\Lambda(x+y,z)=\Lambda(x,z)+\Lambda(y,z)$, for arbitrary $x,y,z\in X$

(b) $\Lambda(A(x,y))=(\det A)\Lambda(x,y)$, for arbitrary $x,y\in X$ and $A\in M_2(\Phi)$.

is called **skew-symmetric 2-linear form** or simply **2-linear functional**.

Completely analogous to the definition of a 2-linear functional, which is essentially a definition of a skew-symmetric 2-form, the definitions of a 2-seminorm and a 2-norm are interchangeable.

Definition 4. Let X be a vector space over the field Φ. The mapping $p: X^2 \to \mathbb{R}$ for which the following conditions hold

(a) $p(x+y,z) \leq p(x,z)+p(y,z)$, for every $x,y,z \in X$

(b) $p(A(x,y)) = |\det A|\, p(x,y)$,for every $x,y \in X$ and $A \in M_2(\Phi)$.

is called a **2-semi norm** and (X^2, p) is called a **2-semi normed space**.

Definition 5. The mapping $\|\cdot\|: X^n \to \mathbb{R}$, $n \geq 2$ for which the following conditions hold:

(a) $\| x_1, x_2 \| = 0$ if and only if x_1, x_2 are linear dependent vectors;

(b) $\| A(x_1,x_2) \| = |\det A| \| x_1, x_2 \|$, for all $x_1, x_2 \in X$ and all $A \in M_2(\Phi)$;

(c) $\| x_1 + x_2, x_3 \| \leq \| x_1, x_3 \| + \| x_2, x_3 \|$, for every $x_1, x_2, x_3 \in X$,

we call **2-norm** of the vector space X, and the ordered pair $(X, \|\cdot,\cdot\|)$ we call **2-normed space**.

In this section, some of the special types of subsets of X^2 will be considered.

Definition 6. The subset S, $S \subseteq X^2$ which is closed with respect to the operations of the 2 -space X^2 is called **2-subspace** of X^2.

Of course in these considerations the following two theorems are important.

Theorem. *The intersection of an arbitrary family of 2-subspaces of the 2-vector space* X^2 *is a 2-subspace.*

According to the last theorem, each subset $A \subseteq X^2$ determines a 2-subspace S_A, the smallest 2-subspace of the 2-vector space X^2 which contains the set A. We will call the 2-subspace S_A the **2-subspace generated by the set** A, and the set A **-the generating set**.

In this matter we will consider a special type of generating sets, i.e. a generating set of the form $M \cup \{(u,v)\}$, where M is a special type of a 2-subspace, and $(u,v) \in X^2$ is arbitrarily given where $\{u,v\}$ is a linearly independent set.

The basic question which we will consider here is whether it is possible to extend a 2-skew-symmetric linear form defined on some types, i.e. classes 2-subspaces to a bigger subspace, in the sense of extension of 2-linear functionals, i.e. of the type of Hanh-Banach. At this moment it will be done only in special cases. The main aim if all such considerations is whether we can prove the following theorem or some of its variants.

Theorem 2. *Let* S *be a 2-subspace of the 2-space* X^2, $\Lambda: S \to \mathbb{R}$ *be 2-skew-symmetric linear form, and* $p: X^2 \to \mathbb{R}$ *be a mapping for which*

(a) $p(x+y,z) \leq p(x,z) + p(y,z)$, *for all* $x,y,z \in X$

(b) $p(tx,y) = tp(x,y)$, *for all* $x,y \in X$ *and* $t > 0$.

There exists 2-skew-symmetric linear form $\Lambda': X^2 \to \mathbb{R}$, *such that* $\Lambda'/S = \Lambda$

Each 2-semi norm satisfies the conditions a) and b) from the previous theorem.

In addition, in many parts we will meet a special type of subsets from X^2. One of them is given by the following definition.

Definition 6'. The subset $T, T \subseteq X^2$ is called n**-invariant** if $AT \subseteq T$ for every $A \in M_2(\Phi)$, $\det A = 1$.

The general structure of 2-subspaces is, of course, not simple. The simplest forms of 2-subspaces are the kernel subspaces, knot subspaces, branch subspaces and cyclic subspaces. Those are discussed and described in [6].

Solving the problem presented in the last theorem is of course not simple. An affirmation of that is of course the complex structure of the 2-subspaces of the 2-space X^2. Due to this, we will discuss partial cases of this problem.

In this paper we will look at extension of 2-skew-symmetric form defined on a two sided branch-2-subspace.

From here on, we will assume that the subset $\{..., x_{-n}, x_{-(n-1)}, x_{-(n-2)}, ..., x_{-1}, x_{\circ}, x_1, x_2, ..., x_n, ...\}$ is a linearly independent subset of the vector space X, not excluding the case when it is finite.

Definition 7. Let X be a vector space over the field Φ. The 2-subspace S generated by the subset
$\{...,(x_{-n}, x_{-(n-1)}),....,(x_{-2}, x_{-1}),(x_{-1}, x_{\circ}),(x_{\circ}, x_1),(x_1, x_2),(x_2, x_3),(x_3, x_4),....,(x_{n-1}, x_n),....\}$
where $\{..., x_{-n}, ..., x_{-2}, x_{-1}, x_{\circ}, x_1, x_2, ..., x_n, ...\}$ is linearly independent set is called a **two- branch** 2**-subspace.**

A detailed description of branch 2-subspaces is given in [7]. That is the content of the theorem that follows.

Theorem 3. *If* M *is a branch 2-subspace generated by the set*
$\{...,(x_{-n}, x_{-(n-1)}),....,(x_{-2}, x_{-1}),(x_{-1}, x_{\circ}),(x_{\circ}, x_1),(x_1, x_2),(x_2, x_3),(x_3, x_4),....,(x_{n-1}, x_n),....\}$
where $\{..., x_{-n}, ..., x_{-2}, x_{-1}, x_{\circ}, x_1, x_2, ..., x_n, ...\}$ *is a linearly independent set, then*

$$M = \bigcup_{i \in \mathbb{Z}} \ \bigcup_{a_{i-1}, a_{i+1} \in \Phi} L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i).$$

In the following part we will consider extension of a two-sided branch 2-subspace M with the addition of one element (u,v) as well as extension of a 2-skew-symmetric form $\Lambda : M \to \mathbb{R}$ to a skew-symmetric form on $\Lambda' : M' \to \mathbb{R}$ where $M' = \langle M \cup \{(u,v)\} \rangle$.

The leading result in the description of the special 2-subspaces such as cyclic, branch 2-subspaces, kernel 2-subspaces and knot 2-subspaces is the following lemma:

Lemma. *The subspace generated by the elements* $(x_{i-1}, x_i),(x_i, x_{i+1}),(x_{i+1}, x_{i+2})$ *i.e.* $L[(x_{i-1}, x_i),(x_i, x_{i+1}),(x_{i+1}, x_{i+2})]$, *where* $\{x_{i-1}, x_i, x_{i+1}, x_{i+2}\}$ *is a linearly independent set is*

$$L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \times L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \cup L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i)$$

The idea for such lemma comes from the fact that it looks like as we have put together two branch 2-subspases which are

$$L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \times L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \qquad (1)$$

and $L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\times L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)$. (2)

Here, they have as their 2-subspace a set defined with

$M=\{(A(x_i,x_{i+1})^T\,/\,A\in M_2(\Phi)\}$.

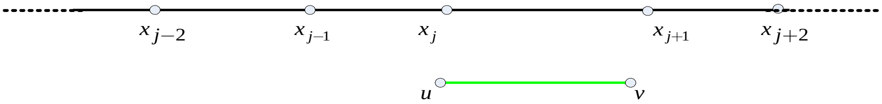
Adding of elements of (1) and (2) of course is possible, but the result is always an element that either belongs in one of these 2-subspaces i.e. either is in (1) or is in (2). If it belongs in both subspaces, then it is an element of the 2-subspace $M=\{(A(x_i,x_{i+1})^T\,/\,A\in M_2(\Phi)\}$.

2. EXTENSION OF A TWO-SIDED BRANCH 2-SUBSPACE

Let Λ be a skew-symmetric linear form defined on a two-sided branch 2-subspace M which is generated by the elements of the set $\{\dots,(x_{-n},x_{-(n-1)}),\dots,(x_{-2},x_{-1}),(x_{-1},x_\circ),(x_\circ,x_1),(x_1,x_2),(x_2,x_3),(x_3,x_4),\dots,(x_{n-1},x_n),\dots\}$ where $\{\dots,x_{-n},\dots,x_{-2},x_{-1},x_\circ,x_1,x_2,\dots,x_n,\dots\}$ is a linearly independent set. Let $(u,v)\in X^2$ be such that $\{u,v\}$ is a linearly independent set. We denote the 2-subspace of X^2 generated by $M\cup\{(u,v)\}$ by M'. Several cases are possible.

Case 1. $u,v\notin L(\dots,x_{-n},\dots,x_{-2},x_{-1},x_\circ,x_1,x_2,\dots,x_n,\dots)$, where $L(\dots,x_{-n},\dots,x_{-2},x_{-1},x_\circ,x_1,x_2,\dots,x_n,\dots)$ is the subspace of X generated by $\{\dots,x_{-n},\dots,x_{-2},x_{-1},x_\circ,x_1,x_2,\dots,x_n,\dots\}$.

The 2-subspace generated by $\{(u,v)\}$ is $L(u,v)\times L(u,v)$. Let us notice that $L(u,v)\cap L(\dots,x_{-n},x_{-(n-1)},\dots,x_o,x_1,\dots)\subset\Delta_2$. Accordingly, $M'=M\cup L(u,v)\times L(u,v)$, where M is determined in theorem 3.



According to our conditions for this part, we have that

$$u=\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}+\dots+\alpha_{i+k}x_{i+k}+\alpha z$$

$$v=\beta_j x_j+\beta_{j+1}x_{j+1}+\beta_{j+2}x_{j+2}+\dots+\beta_{j+p}x_{j+p}+\beta w\text{’}$$

where $x,y\notin L(\dots,x_{-n},\dots,x_{-2},x_{-1},x_\circ,x_1,x_2,\dots,x_n,\dots);i,j,k,p\in\mathbb{Z}$ are given arbitrary. In that case, the vector (u,v) cannot add with the elements of the set M at any case. Indeed, we can write the elements u and v in the form $u=x+\alpha z, v=y+\beta w$, i.e. $(u,v)=(x+\alpha z,y+\beta w)$. For any element $(x',y')\in M$ addition is not possible, because $u\neq x', v\neq y'$. From the other hand, for any nonsingular matrix $A=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}$, we have the following situations (all possible cases will be considered).

Situation 1. $a_{11}\neq 0$, $a_{12}=0$.

In this sub case we have the following three possibilities:

a) $a_{21}\neq 0$, $a_{22}=0$, which is not possible, because in this situation we would have that $\det A=0$, which is not possible.

b) $a_{21}=0$, $a_{22}\neq 0$, which is possible. In this situation $\det A = a_{11}a_{22}\neq 0$. Here

$$A(u,v)^T=\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix} u\\ v\end{bmatrix}=(a_{11}u,a_{22}v)=(a_{11}(x+\gamma y),a_{22}(w+\delta z))=(a_{11}x+a_{11}\gamma y,a_{22}w+a_{22}\delta z)$$

where form because of the condition $a_{11}\gamma a_{22}\delta\neq 0$, we get that $A(u,v)^T\not\in M$. This element will belong in the new set in the part where it is added.

c) $a_{21}\neq 0$, $a_{22}\neq 0$, which is possible. In this situation $\det A = a_{11}a_{22}\neq 0$. Here, as in the previous case

$$A(u,v)^T=\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix} u\\ v\end{bmatrix}=(a_{11}u,a_{21}u+a_{22}v)=(a_{11}(x+\gamma y),a_{21}(x+\gamma y)+a_{22}(w+\delta z))=$$

$$=(a_{11}x+a_{11}\gamma y,a_{21}x+a_{22}w+a_{21}\gamma y+a_{22}\delta z)$$

where form because of the condition $a_{11}\gamma\neq 0$ we have that $A(u,v)^T\not\in M$. This element will belong in the new set $\left\{A(u,v)^T\,/\,A\in M_2(\Phi)\right\}$.

Situation 2. $a_{11}=0$, $a_{12}\neq 0$

In this sub case we have the following three possibilities:

a) $a_{21}\neq 0$, $a_{22}=0$, which is possible, because in this case we have

$$\det A=\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}=-a_{12}a_{21}\neq 0\,.$$

But, in this case we have that

$$A(u,v)^T=\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix} u\\ v\end{bmatrix}=(a_{12}v,a_{21}u)=(a_{12}(w+\delta z),a_{21}(x+\gamma y))=(a_{12}w+a_{12}\delta z,a_{21}x+a_{21}\gamma y)$$

and because of the condition $a_{12}\delta a_{21}\gamma\neq 0$, $A(u,v)^T\not\in M$

b) $a_{21}=0$, $a_{22}\neq 0$, which is possible from technical aspect. But, as

$$\det A=\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}=0\cdot a_{22}-0\cdot a_{21}=0\,,$$

and by the conditions we have that $\det A\neq 0$. Because the contradiction, this case is not possible in this situation.

c) $a_{21}\neq 0$, $a_{22}\neq 0$. This case is possible from technical aspect. Indeed,

$$\det A=\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}=0\cdot a_{22}-a_{12}\cdot a_{21}=-a_{12}\cdot a_{21}\neq 0\,.$$

In this situation, we have

$$A(u,v)^T=\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix} u\\ v\end{bmatrix}=(a_{12}v,a_{21}u+a_{22}v)=(a_{12}(w+\delta z),a_{21}(x+\gamma y)+a_{22}(w+\delta z))=$$

$$=(a_{12}w+a_{12}\delta z,a_{21}x+a_{21}\gamma y+a_{22}w+a_{22}\delta z)=$$

$$=a_{12}(w+\delta z,a_{21}x+a_{21}\gamma y+a_{22}w+a_{22}\delta z)\notin M$$

because the first component $w+\delta z\in L(x_1,x_2,x_3,x_4)$.

Situation 3. $a_{11}\neq 0$, $a_{12}\neq 0$.

In this sub case we have the following three possibilities:

a) $a_{21} \neq 0$, $a_{22} = 0$, which is possible, because in this case we have

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -a_{21}a_{12} \neq 0,$$

and the matrix is nonsingular. According to this,

$$A(u,v)^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = (a_{12}v, a_{21}u) = (a_{12}(w+\delta z), a_{21}(x+\gamma y)) = a_{12}a_{21}(w + a_{12}\delta z, x+\gamma y) \notin M$$

b) $a_{21} = 0$, $a_{22} \neq 0$, which is also possible, where

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} \neq 0. \text{ Here}$$

$$A(u,v)^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = (a_{11}u, a_{22}v) = (a_{11}(x+\gamma y), a_{22}(w+\delta z)) = a_{11}a_{22}(x+\gamma y, w + a_{12}\delta z) \notin M$$

c) $a_{21} \neq 0$, $a_{22} \neq 0$. Because of its nature, this case is the most radical one. Here, if we use the technique from that we have in the 2-normed spaces, we have

$$\begin{aligned} A(u,v)^T &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = (a_{11}u + a_{12}v, a_{21}u + a_{22}v) = \\ &= (a_{11}(x+\gamma y) + a_{12}(w+\delta z), a_{21}(x+\gamma y) + a_{22}(w+\delta z)) = \\ &= (a_{11}x + a_{11}\gamma y + a_{12}w + a_{12}\delta z, a_{21}x + a_{22}w + a_{21}\gamma y + a_{22}\delta z) \sim (=) \\ &= (\det A)(x+\gamma y, w+\delta z) = ((\det A)x + (\det A)\gamma y, w+\delta z) \notin M \end{aligned}$$

and because $(\det A)\gamma \neq 0$, where from we get that the first element doesn't belong at any element of M and so, the whole element doesn't belong in M. Also, let us comment that in the part where we have $\sim (=)$ we have a sign for equality. But, that is not a problem, because from that element until the last element we constantly multiply with a matrix that has a determinant equal to 1, and because of this if one element doesn't belong in M, then any other multiplied with a matrix with determinant equal to 1 does not belong in M. The last equality may not be used, because $\det A \neq 0$ и $\gamma \neq 0$ where from follows the proof.

For this case the extension is arbitrary, i.e. $\Lambda'(u,v) = \alpha$, where α is an arbitrary fixed scalar.

Case 2. Let $u \in L(\ldots, x_{-n}, \ldots, x_{-2}, x_{-1}, x_{\circ}, x_1, x_2, \ldots, x_n, \ldots)$ and $v \notin L(\ldots, x_{-n}, \ldots, x_{-2}, x_{-1}, x_{\circ}, x_1, x_2, \ldots, x_n, \ldots)$

In this case we will consider several sub cases, as follows.

Sub case 1. $u = x_i$ for some $i \in \mathbb{N}$.

In this sub case the set $\{(x_{i-1}, x_i), (x_i, x_{i+1}), (x_i, v)\} = \{(x_{i-1}, u), (u, x_{i+1}), (u, v)\}$ generates a 2-subspace which is a knot subspace and its form is

$$L = \bigcup_{w \in L(x_{i-1}, v, x_{i+1})} L(u,w) \times L(u,w).$$

Simultaneously the sets $P' = \{\ldots., (x_{i-6}, x_{i-5}), (x_{i-5}, x_{i-4}), \ldots., (x_{i-2}, x_{i-1})\}$ and $P'' = \{(x_{i+1}, x_{i+2}), (x_{i+2}, x_{i+3}), \ldots, (x_{m-1}, x_m), \ldots\}$ generate 2-subspaces $S_{P'}$ and $S_{P''}$

respectively, which are one-sided branch 2-subspaces. At the same time, they, as well as L , are 2 -subspaces from the required extension M'. The forms of $S_{P'}$ and $S_{P''}$ are

$$S_{P'}=\bigcup_{k=-\infty}^{i-1}\ \bigcup_{a_{k-1},a_{k+1}\in\Phi} L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)\times L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)$$

$$S_{P''}=\bigcup_{k=i+1}^{\infty}\ \bigcup_{a_{k-1},a_{k+1}\in\Phi} L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)\times L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)$$

In order for us to see the form of M' it is enough to consider several types of addition of elements of $L,S_{P'}$ and $S_{P''}$. It is enough to consider several cases.

1° $(m,n)\in L$, $\quad (x,y)\in L((x_{i-2},x_{i-1}),(x_{i-1},x_i))$

2° $(m,n)\in L$, $\quad (x,y)\in L((x_{i-3},x_{i-2}),(x_{i-2},x_{i-1}))$

3° $(m,n)\in L$, $\quad (x,y)\in L((x_i,x_{i+1}),(x_{i+1},x_{i+2}))$

4° $(m,n)\in L$, $\quad (x,y)\in L((x_{i+1},x_{i+2}),(x_{i+2},x_{i+3}))$.

In case 1° we have

$$(m,n)=(b_1(\alpha_1x_{i-1}+\alpha_2v+\alpha_3x_{i+1})+b_2x_i,b_3(\alpha_1x_{i-1}+\alpha_2v+\alpha_3x_{i+1})+b_4x_i)$$

$$(x,y)=(a_1(\alpha x_{i-2}+\beta x_i)+a_2x_{i-1},a_3(\alpha x_{i-2}+\beta x_i)+a_4x_{i-1})\ .$$

In other words, the following elements should add

$(x,y)=(a_1\alpha x_{i-2}+a_2x_{i-1}+a_1\beta x_i,a_3\alpha x_{i-2}+a_4x_{i-1}+a_3\beta x_i)$ and

$$(m,n)=(b_1\alpha_1x_{i-1}+b_2x_i+b_1\alpha_2v+b_1\alpha_3x_{i+1},b_3\alpha_1x_{i-1}+b_4x_i+b_3\alpha_2v+b_3\alpha_3x_{i+1})$$

Since $\{x_{i-2},x_{i-1},x_i\}$ and $\{x_{i-1},x_i,x_{i+1},v\}$ are linearly independent sets, that is possible only in the case when

a) $\alpha_2=\alpha_3=\alpha=0$, $b_1\alpha_1=a_2=s$, $a_1\beta=b_2=t$, or

b) $\alpha_2=\alpha_3=\alpha=0$, $b_3\alpha_1=a_4=s$, $a_3\beta=b_4=t$,

for any $s,t\in\mathbf{\Phi}$ (the cases when $t=0$ or $s=0$ or when $s=t=0$ should be considered separately).

In case a) the elements get the form

$$(b_1\alpha_1x_{i-1}+b_2x_i,b_3\alpha_1x_{i-1}+b_4x_i)=(sx_{i-1}+tx_i,b_3\alpha_1x_{i-1}+b_4x_i)$$

$$(a_1\beta x_i+a_2x_{i-1},a_3\beta x_i+a_4x_{i-1})=(sx_{i-1}+tx_i,a_3\beta x_i+a_4x_{i-1})\ ,$$

and their sum is

$$(sx_{i-1}+tx_i,(a_3\beta+b_4)x_i+(a_4+b_3\alpha_1)x_{i-1})\in L((x_{i-1},x_i))\subset L$$

We similarly get for case b).

In case 2° we have

$$(x,y)=(a_1(\alpha x_{i-3}+\beta x_{i-1})+a_2x_{i-2},a_3(\alpha x_{i-3}+\beta x_{i-1})+a_4x_{i-2})$$

$$(m,n)=(b_1(\alpha_1x_{i-1}+\alpha_2v+\alpha_3x_{i+1})+b_2x_i,b_3(\alpha_1x_{i-1}+\alpha_2v+\alpha_3x_{i+1})+b_4x_i)$$

Similarly as in 1° we have to add the both elements

$(x,y)=(a_1\alpha x_{i-3}+a_2x_{i-2}+a_1\beta x_{i-1},a_3\alpha x_{i-3}+a_4x_{i-2}+a_3\beta x_{i-1})$ and

$$(m,n)=(b_1\alpha_1x_{i-1}+b_1\alpha_2v+b_2x_i+b_1\alpha_3x_{i+1},b_3\alpha_1x_{i-1}+b_3\alpha_2v+b_4x_i+b_3\alpha_3x_{i+1})$$

Since $\{x_{i-3},x_{i-2},x_{i-1}\}$ and $\{x_{i-1},x_i,x_{i+1},v\}$ are linearly independent sets, that is possible only in one of the following two situations:

c) $\alpha_2 = \alpha_3 = \alpha = 0$, $a_2 = b_2 = 0$, $a_1\beta = b_1\alpha_1 = s$

d) $\alpha_2 = \alpha_3 = \alpha = 0$, $a_4 = b_4 = 0$, $a_3\beta = b_3\alpha_1 = s$,

and for every conditions for arbitrary $s \in \mathbf{\Phi}$ (for the same conditions for $s = 0$ should be considered separately).

In case c) the elements get the form

$$(sx_{i-1}, a_3\beta x_{i-1} + a_4 x_{i-2})$$

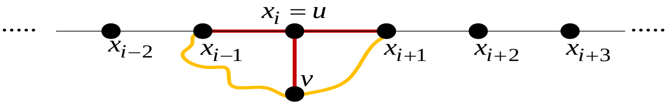
$$(sx_{i-1}, b_3\alpha_1 x_{i-1} + b_4 x_i)$$

and their sum is $(sx_{i-1}, (a_3\beta + b_3\alpha_1)x_{i-1} + a_4 x_{i-2} + b_4 x_i) \in L((x_{i-2}, x_{i-1}), (x_{i-1}, x_i)) \subset M$

We similarly get for case d).

According to that, in this sub case the extension is

$$M' = M \cup \bigcup_{w \in L(x_{i-1}, x_i, x_{i+1})} L(x_i, w) \times L(x_i, w).$$



Situation 3° is completely analogue to situation 1°, and the situation 4° is completely analogue to situation 2° (in both situations we are considering the case under the same conditions, but from the opposite side).

Sub case 2. $u \in L(x_j, x_{j+1})$ for some $j \in \mathbb{N}$, where $u \neq x_j, x_{j+1}$.

In this sub case we have $u = \mu x_j + \nu x_{j+1}$, where $\mu\nu \neq 0$. The sets $\{v, u, x_j\}$ and $\{v, u, x_{j+1}\}$ are linearly independent sets. The sets $K' = \{(u,v),(u,x_j)\}$ and $K' = \{(u,v),(u,x_{j+1})\}$ generate 2-subspaces $S_{K'}$ and $S_{K''}$ and their forms are

$$S_{K'} = \bigcup_{\alpha,\beta \in \Phi} L(\alpha v + \beta x_j, u) \times L(\alpha v + \beta x_j, u)$$

$$S_{K''} = \bigcup_{\alpha,\beta \in \Phi} L(\alpha v + \beta x_{j+1}, u) \times L(\alpha v + \beta x_j, u)$$

The general form of the elements of $S_{K'}$ is

$$(a_1(\alpha v + \beta x_j) + a_2 u, a_3(\alpha v + \beta x_j) + a_4 u)$$

and of the elements of $S_{K''}$ is

$$(b_1(\gamma v + \delta x_{j+1}) + b_2 u, b_3(\gamma v + \delta x_{j+1}) + b_4 u).$$

We should note that both sets $\{u, x_{j+1}, v\}$ and $\{x_j, u, v\}$ separately are linearly independent sets. Here the same elements can be written in the following form:

$(a_1\alpha v + a_1\beta x_j + a_2 u, a_3\alpha v + a_3\beta x_j + a_4 u)$ and

$(b_1\gamma v + b_1\delta x_{j+1} + b_2 u, b_3\gamma v + b_3\delta x_{j+1} + b_4 u)$.

Addition of the latter two forms of elements is possible in the following 2 cases:

a) $\beta = \delta = 0$, $a_2 = b_2 = t$, $a_1\alpha = b_1\gamma = s$

or b) $\beta = \delta = 0$, $a_2 = b_2 = t$, $a_3\alpha = b_3\gamma = s$.

In case a) the elements get the form

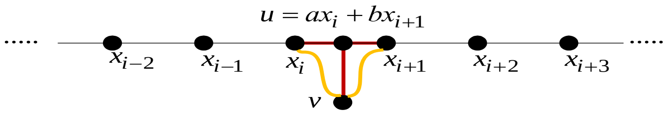
$(sv+tu, a_3\alpha v+a_4u)$
$(sv+tu, b_3\gamma v+b_4u)$

and their sum is $(sv+tu,(b_3\gamma+a_3\alpha)v+(a_4+b_4)u)\in L((u,v))\subset M'$

The result in case b) is similar.

From the whole of the former discussion it is clear that $M'=M\cup S_{K'}\cup S_{K''}$.

We consider the sub cases 3 and 4 similarly.



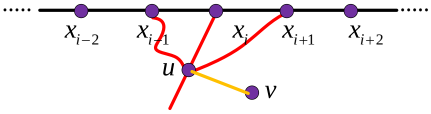
Sub case 3. $u\in L(x_i,x_{i+1},x_{i+2},x_{i+3})$, and the coefficients in the representation before x_i and x_{i+3} are different from zero. In other words, $u=\alpha_ix_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}+\alpha_{i+3}x_{i+3}$, where $\alpha_i\alpha_{i+3}\neq 0$. Such element, as coordinate in the elements of the 2-subspace M doesn't exist, so this case of addition is not possible. If $\alpha_i=0$ or $\alpha_{i+3}=0$, then we come to situation which is in the sub case 4 of this case, or sub case 2 from this case.

Sub case 4. $u\in L(x_i,x_{i+1},x_{i+2})$. Such case is possible because the vector u has the following form $u=\alpha_ix_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$. Here we can take that $u\in L(\alpha_ix_i+\alpha_{i+2}x_{i+2},x_{i+1})$. From the other side, the element v can be any element from the vector space X (see drawing). Here, in order not to disturb the generality, in the most general case we must consider that $u=\alpha_ix_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$ and $v=x+\alpha y$, where $\alpha\neq 0$, y is a nonzero vector from X, and x is from the subspace generated from the vectors which form the 2-subspace, i.e. x_i,x_{i+1},x_{i+2}. The vector cannot be a coordinate of any 2-vector from M. According to this, $(u,v)=(u,x+\alpha y)$ cannot be a 2-vector from M. From the other side, for the vector u we can say that it is obtained as follows:

$$\begin{bmatrix}1&\alpha_{i+1}\\0&1\end{bmatrix}\left(\begin{bmatrix}\alpha_i&0\\0&1\end{bmatrix}(x_i,x_{i+1})+\begin{bmatrix}\alpha_{i+2}&0\\0&1\end{bmatrix}(x_{i+2},x_{i+1})\right)=\begin{bmatrix}1&\alpha_{i+1}\\0&1\end{bmatrix}((\alpha_ix_i,x_{i+1})+(\alpha_{i+2}x_{i+2},x_{i+1}))$$

$$=\begin{bmatrix}1&\alpha_{i+1}\\0&1\end{bmatrix}(\alpha_ix_i+\alpha_{i+2}x_{i+2},x_{i+1})=(\underbrace{\alpha_ix_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}}_{u},x_{i+1})$$

Here we can note that the element u can be obtained also as addition of the elements (x_i,x_{i+2}) and (x_{i+1},x_{i+2}), in exactly the same way as before, but in that case we would get that the elements (x_i,x_{i+2}), (x_{i+1},x_{i+2}), (x_{i+1},x_i) are elements which generate M, and with that, the kernel subspace $S=L(x_i,x_{i+1},x_{i+2})\times L(x_i,x_{i+1},x_{i+2})$ would be a subspace of M, which is not possible. Completely analogous would be the considerations the generating elements to be (x_{i+1},x_i) and (x_{i+2},x_i), which will take us to the same conclusion.



Let's note that we have three possibilities which imply in this situation, i.e.

a) $\alpha_{i+1} = 0$, $a_i a_{i+2} \neq 0$

b) $a_i = 0,\quad a_{i+1}a_{i+2} \neq 0$,

c) $a_{i+2} = 0,\quad a_i a_{i+1} \neq 0$

Situation b)

It is clear that the element u belongs in $L(x_i, x_{i+1})$ which is completely the same with the sub case 2 of this case and will be not considered here.

Situation b)

It is clear that the element u belongs in $L(x_{i+2}, x_{i+1})$, which again is completely the same with the sub case 2 of this case and will be not considered here.

Situation a)

In this situation $u \in L(x_i, x_{i+2})$, and this is element from the set generated from (x_i, x_{i+1}) and (x_{i+1}, x_{i+2}) and the element v is not a coordinate of 2-vector from M. But, now, it is clear that the element $u = \alpha_i x_i + \alpha_{i+2}x_{i+2}$, and here for example belongs in the 2-subspace M' and we have it as a coordinate of the 2-vector $(u, x_{i+1}) = (\alpha_i x_i + \alpha_{i+2}x_{i+2}, x_{i+1})$.

Let's assume that $(u, x_{i+2}) \in M'$. Then

$$M' \ni (u, x_{i+1}) = (\alpha_i x_i + \alpha_{i+2}x_{i+2}, x_{i+2}) = (\alpha_i x_i + \alpha_{i+2}x_{i+2} - \alpha_{i+2}x_{i+2}, x_{i+2}) = (\alpha_i x_i, x_{i+2}) \sim (x_i, x_{i+2})$$

The same discussion goes for $(u, x_i) \in M'$. But, then we get that the 2-subspace generated by (x_i, x_{i+2}), (x_{i+1}, x_{i+2}), (x_{i+1}, x_i), which is a kernel 2-subspace, would be 2-subspace of M', which certainly is not possible.

Now, if we consider addition of two elements of this 2-subspace, then we would have that we can add the 2-vectors

$$(a_{11}(\alpha_i x_i + \alpha_{i+2}x_{i+2}) + a_{12}\alpha_{i+1}x_{i+1}, a_{21}(\alpha_i x_i + \alpha_{i+2}x_{i+2}) + a_{22}\alpha_{i+1}x_{i+1}),$$

$$(b_{11}(\alpha_i x_i + \alpha_{i+2}x_{i+2}) + b_{12}\alpha_{i+1}x_{i+1}, b_{21}(\alpha_i x_i + \alpha_{i+2}x_{i+2}) + b_{22}\alpha_{i+1}x_{i+1})$$

In this case, if for example the second coordinates are equal, then adding the first coordinates we would get the 2-vector

$$((a_{11}+b_{11})(\alpha_i x_i + \alpha_{i+2}x_{i+2}) + (a_{12}+b_{12})\alpha_{i+1}x_{i+1}, a_{21}(\alpha_i x_i + \alpha_{i+2}x_{i+2}) + a_{22}\alpha_{i+1}x_{i+1})$$

i.e. it is the same as we have multiplied with a matrix in the following form $\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21} & a_{22} \end{bmatrix}$, and we would get again 2-vector with the same form.

Finally, we have

$$M' = M \cup \bigcup_{w \in L(x_{i+1}, v)} L(w,u) \times L(w,u).$$

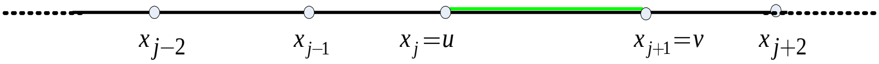
The case $u \notin L(\ldots, x_{-n}, \ldots, x_{-2}, x_{-1}, x_{\circ}, x_1, x_2, \ldots, x_n, \ldots)$ and $v \in L(\ldots, x_{-n}, \ldots, x_{-2}, x_{-1}, x_{\circ}, x_1, x_2, \ldots, x_n, \ldots)$ is completely analogously considered.

Case 3. Let $u,v \in L(...,x_{-n},...,x_{-2},x_{-1},x_{\circ},x_1,x_2,...,x_n,...)$.

We will consider several possibilities, i.e. sub cases.

Sub case 1. $u = x_i$, $v = x_{i+1}$

In this sub case $L(u,v) = L(x_j, x_{j+1})$, therefore we don't have a true extension of M. That is because the 2-vector (u,v) is a 2-vector both in M and in M'. So, in this case $M = M'$.



Sub case 2. $u = x_i$, $v = x_{i+2}$

In this sub case, the pairs (x_i, x_{i+1}), (x_{i+1}, x_{i+2}) and (x_i, x_{i+2}) are included in the generating of M' so, accordingly, they define a kernel subspace S which is of the form $L(x_i, x_{i+1}, x_{i+2}) \times L(x_i, x_{i+1}, x_{i+2})$. Now, the subspace M' is generated by one kernel subspace S, and two branch 2-subspaces, one generated by $....,(x_{i-2},x_{i-1}),(x_{i-1},x_i)$ and the other by $(x_{i+2},x_{i+3}),(x_{i+3},x_{i+4}),....,(x_m,x_{m+1}),(x_{m+1},x_{m+2}),.....$

The form of S is $S = L(x_i, x_{i+1}, x_{i+2}) \times L(x_i, x_{i+1}, x_{i+2})$.

The form of the 2-subspace S' is

$$S' = \bigcup_{k=-\infty}^{i-1} \bigcup_{a_{k-1},a_{k+1}\in\Phi} L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k) \times L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k)$$

The form of the 2-subspace S'' is

$$S'' = \bigcup_{k=i+3}^{\infty} \bigcup_{a_{k-1},a_{k+1}\in\Phi} L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k) \times L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k)$$

Let us notice that the addition of elements of S or S' or S'' is again an element of S or S' or S'', respectively. Addition of elements of S' and S'', one from S' and the other from S'', is not possible.

We will determine when addition of elements of S and S' is possible and what is the result of that addition. Every element of S is of the form

$$(a_1x_i + b_1x_{i+1} + c_1x_{i+2}, a_2x_i + b_2x_{i+1} + c_2x_{i+2})$$

and the elements from S' for which addition is possible are of the form

$$(d_1(\alpha x_{i-2} + \beta x_i) + e_1x_{i-1}, d_2(\alpha x_{i-2} + \beta x_i) + e_2x_{i-1}).$$

Addition in this case is possible in the following two cases:

a) $b_1 = c_1 = 0$, $\alpha = 0$, $d_1\beta = a_1 = s$

b) $b_2 = c_2 = 0$, $\alpha = 0$, $d_2\beta = a_2$.

It is enough to consider the case a). Then the elements obtain the form

$$(sx_i, a_2x_i + b_2x_{i+1} + c_2x_{i+2}),\ (sx_i, d_2\beta x_i + e_2x_{i-1})$$

and their sum is

$$(sx_i, (a_2 + d_2\beta)x_i + b_2x_{i+1} + c_2x_{i+2} + e_2x_{i-1}).$$

Therefore, the sum of these elements is an element from the 2-subspace T defined by

$$T = \bigcup_{u \in L(x_{i-1}, x_{i+1}, x_{i+2})} L(x_1, u) \times L(x_1, u) .$$

Now it is enough to determine the sum of the elements from the 2-subspace T with the elements of the 2-subspace generated by the elements of the set $\{(x_{i-3}, x_{i-2}), (x_{i-2}, x_{i-1})\}$. The former are of the form

$$\begin{aligned} &A(x_i, \alpha_1 x_{i-1} + \alpha_2 x_i + \alpha_3 x_{i+1} + \alpha_4 x_{i+2}) = \\ &\quad = (b_1 x_i + b_2(\alpha_1 x_{i-1} + \alpha_2 x_i + \alpha_3 x_{i+1} + \alpha_4 x_{i+2}), b_3 x_i + b_4(\alpha_1 x_{i-1} + \alpha_2 x_i + \alpha_3 x_{i+1} + \alpha_4 x_{i+2})) \end{aligned} \qquad (*)$$

The subspace generated by the set $\{(x_{i-3}, x_{i-2}), (x_{i-2}, x_{i-1})\}$ is

$$\bigcup_{\alpha, \beta \in \Phi} L(\alpha x_{i-3} + \beta x_{i-1}, x_{i-2}) \times L(\alpha x_{i-3} + \beta x_{i-1}, x_{i-2}) ,$$

and its elements are of the form

$$(a_1(\alpha x_{i-3} + \beta x_{i-1}) + a_2 x_{i-2}, a_3(\alpha x_{i-3} + \beta x_{i-1}) + a_4 x_{i-2}) . \qquad (**)$$

Elements of the form (*) and (**) is feasible in two cases:

c) $b_1 = 0$, $\alpha_2 = \alpha_3 = \alpha_4 = 0$, $\alpha = 0$, $a_2 = 0$, $b_2 \alpha_1 = a_1 \beta = s$

d) $b_3 = 0$, $\alpha_2 = \alpha_3 = \alpha_4 = 0$, $\alpha = 0$, $a_4 = 0$, $b_4 \alpha_1 = a_3 \beta = s$.

In the case c) we have

$(s x_i, b_3 x_i + b_4 \alpha_1 x_{i-1})$

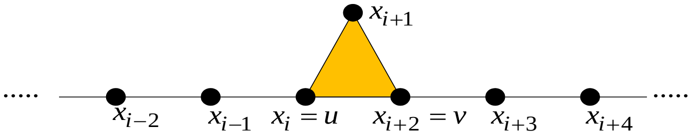
$(s x_i, a_3 \beta x_{i-1} + a_4 x_{i-2})$

and their sum is

$$(s x_i, b_3 x_i + (b_4 \alpha_1 + a_3 \beta) x_{i-1} + a_4 x_{i-2}) \in L(\gamma x_{i-2} + \delta x_i, x_{i-1}) \times L(\gamma x_{i-2} + \delta x_i, x_{i-1}) .$$

The case d) is considered analogously.

Accordingly, in this case M' is the 2-subspace



$$M' = M \cup (L(x_i, x_{i+1}, x_{i+2}))^2 \cup \bigcup_{u \in L(x_{i-1}, x_{i+1}, x_{i+2})} L(u, x_i) \times L(u, x_i) \cup \bigcup_{v \in L(x_{i+3}, x_{i+1}, x_i)} L(v, x_{i+2}) \times L(v, x_{i+2})$$

Sub case 3. $u = x_i$, $v = x_j$, $j > i + 2$, for any such j, which is arbitrary, but fixed.

In this sub case the ordered pairs $(x_i, x_{i+1}), (x_{i+1}, x_{i+2}), \ldots., (x_{j-1}, x_j), (x_j, x_i)$ form a cyclic subspace S. Now, the extension is generated by one cyclic subspace S, and two branch 2-subspaces, one S' generated by $\ldots, (x_{i-2}, x_{i-1}), (x_{i-1}, x_i)$ and the other S'' generated by $(x_j, x_{j+1}), (x_{j+1}, x_{j+2}), \ldots., (x_m, x_{m+1}), (x_{m+1}, x_{m+2}), \ldots.$.

The form of S is: $S = \bigcup_{i=1}^{n} [L(a_{i+1} x_{i+1} + a_{i-1} x_{i-1}, x_i) \times L(a_{i+1} x_{i+1} + a_{i-1} x_{i-1}, x_i)]$.

The form of S' is: $S' = \bigcup_{k=-\infty}^{i-1} \bigcup_{a_{k-1}, a_{k+1} \in \Phi} L(a_{k-1} x_{k-1} + a_{k+1} x_{k+1}, x_k) \times L(a_{k-1} x_{k-1} + a_{k+1} x_{k+1}, x_k)$

The form of S'' is: $S''=\bigcup\limits_{k=j+1}^{\infty}\bigcup\limits_{a_{k-1},a_{k+1}\in\Phi}L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)\times L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)$

Addition of elements of S' and S'', i.e. one element from S' and the other from S'' is not possible.

We will consider the remaining possibilities for addition of elements of S,S' and S''. Let us notice that the sets $K'=\{(x_{i-1},x_i),(x_i,x_{i+1}),(x_i,x_j)\}$ and $K''=\{(x_i,x_j),(x_{j-1},x_j),(x_j,x_{j+1})\}$ are generators of the 2-subspaces S_K and $S_{K'}$ which are subspaces of M'. At the same time they are loop 2-subspaces generated by three elements. We have:

$$S_{K'}=\bigcup_{u\in L(x_i,x_{j-1},x_{j+1})}L(u,x_j)\times L(u,x_j) \text{ and } S_{K''}=\bigcup_{v\in L(x_{i-1},x_{i+1},x_j)}L(v,x_i)\times L(v,x_i) .$$

First we will determine when addition is possible between elements from $S_{K'}$ and $S_{K''}$ and what will the result from the addition be. The elements from $S_{K'}$ are of the form

$$(a_1(\alpha_1x_i+\alpha_2x_{j-1}+\alpha_3x_{j+1})+b_1x_j,a_2(\alpha_1x_i+\alpha_2x_{j-1}+\alpha_3x_{j+1})+b_2x_j)$$

and the elements of $S_{K''}$ are of the form

$$(c_1(\beta_1x_{i-1}+\beta_2x_{i+1}+\beta_3x_j)+d_1x_i,c_2(\beta_1x_{i-1}+\beta_2x_{i+1}+\beta_3x_j)+d_2x_i).$$

It is clear that addition is possible in two cases:

a) $\alpha_2=\alpha_3=\beta_1=\beta_2=0$, $a_1\alpha_1=d_1=s$, $c_1\beta_3=b_1=t$

b) $\alpha_2=\alpha_3=\beta_1=\beta_2=0$, $a_2\alpha_1=d_2=s$, $c_2\beta_3=b_2=t$.

In case a) we have the sum

$$(sx_i+tx_j,(a_2\alpha_1+d_2)x_i+(c_2\beta_3+b_2)x_j)\in L((x_i,x_j))$$

In case b) we have the sum

$$((a_1\alpha_1+d_1)x_i+(c_1\beta_3+b_1)x_j,sx_i+tx_j)\in L((x_i,x_j))$$

Therefore in each case the sum is an element from the 2-subspace $L((x_i,x_j))$

We will determine the sums in the remaining possibilities for addition in M' We have the following possibilities:

1° $(x,y)\in S_{K''}$ and $(m,n)\in L((x_{i-3},x_{i-2}),(x_{i-2},x_{i-1}))$

2° $(x,y)\in S_{K''}$ and $(m,n)\in L((x_{i+1},x_{i+2}),(x_{i+2},x_{i+3}))$

3° $(x,y)\in S_{K'}$ and $(m,n)\in L((x_{j-3},x_{j-2}),(x_{j-2},x_{j-1}))$

4° $(x,y)\in S_{K'}$ and $(m,n)\in L((x_{j+3},x_{j+2}),(x_{j+2},x_{j+1}))$

In 1° the elements from $S_{K''}$ are of the form

$$(c_1(\beta_1x_{i-1}+\beta_2x_{i+1}+\beta_3x_j)+d_1x_i,c_2(\beta_1x_{i-1}+\beta_2x_{i+1}+\beta_3x_j)+d_2x_i \text{ and}$$

$$(m,n)=(a_1(\alpha x_{i-3}+\beta x_{i-1})+b_1x_{i-2},a_2(\alpha x_{i-3}+\beta x_{i-1})+b_2x_{i-2})).$$

Therefore, addition is possible in the following two cases:

c) $\beta_2=\beta_3=0$, $\alpha=0$, $b_1=0$, $d_1=0$, $c_1\beta_1=a_1\beta=t$

d) $\beta_2=\beta_3=0$, $\alpha=0$, $b_1=0$, $d_1=0$, $c_2\beta_1=a_2\beta=t$

In the case c) we get

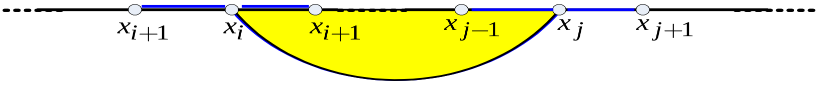
$$(tx_{i-1}, c_2\beta_1 x_{i-1} + d_2 x_i)$$
$$(tx_{i-1}, a_2\beta x_{i-1} + b_2 x_{i-2})$$

and for the sum we get

$$(tx_{i-1}, (c_2\beta_1 + a_2\beta)x_{i-1} + d_2 x_i + b_2 x_{i-2}) \in L((x_{i-2}, x_{i-1}), (x_{i-1}, x_i))$$

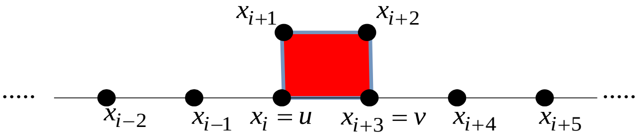
The case d) can be analogously considered.

Similar results are obtained in 2°, 3° and 4° with the results of the additions being elements of the 2-subspaces $L((x_i, x_{i+1}), (x_{i+1}, x_{i+2}))$, $L((x_{j-2}, x_{j-1}), (x_{j-1}, x_j))$ and $L((x_{j+2}, x_{j+1}), (x_{j+1}, x_j))$ respectively, and also being elements of M .



The remaining cases for addition, when it is possible, are addition of elements M and they again belong to M .

Finally, we can conclude that in this sub case:



$$M' = M \cup \bigcup_{u \in L(x_{i-1}, x_{i+1}, x_j)} L(u, x_i) \times L(u, x_i) \cup \bigcup_{v \in L(x_{j+1}, x_i, x_{j-1})} L(v, x_j) \times L(v, x_j) \cup S' \cup S'' \cup S$$

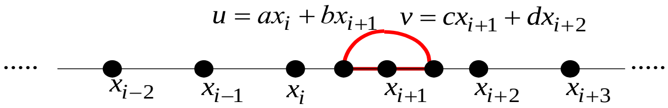
Sub case 5. $u = x_i$, $v = x_j$ where $j = i+3$, i.e. $u = x_i$ and $v = x_{i+3}$.

In this sub case we have that the 2-vectors $(u = x_i, x_{i+1}), (x_{i+1}, x_{i+2}), (x_{i+2}, x_{i+3} = v), (v = x_{i+3}, u = x_{i+1})$ make a cyclic 2-subspace. According to the previous sub case, we have that

$$M' = M \cup S \cup S' \cup S'' \cup K' \cup K'',$$

where K' and K'' are loop 2-subspaces with loop centers u and v, and S' is the branch generated from the elements $\dots, x_{i-3}, x_{i-2}, x_{i-1}$, S'' is a branch 2-subspase generated from $x_{i+4}, x_{i+5}, x_{i+6}, \dots$ and S is a cyclic 2-subspace generated from $(u = x_i, x_{i+1}), (x_{i+1}, x_{i+2}), (x_{i+2}, x_{i+3} = v), (v = x_{i+3}, u = x_{i+1})$.

Sub case 6. $u = ax_i + bx_{i+1}$, $v = cx_{i+1} + dx_{i+2}$ where $ab \neq 0$ and $cd \neq 0$.



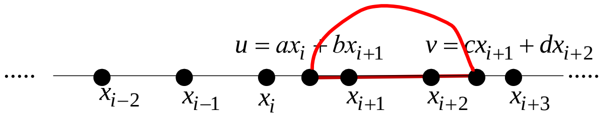
In this case we have that the 2-vectors (v, u) and (x_{i+1}, u) belong in the new 2-subspace M', so according to this in this 2-subspace belongs also the 2-vector

$$\begin{bmatrix} \frac{1}{d} & 0 \\ 0 & 1 \end{bmatrix} \left((v, u) + \begin{bmatrix} -c & 0 \\ 0 & 1 \end{bmatrix} (x_{i+1}, u) \right) = \begin{bmatrix} \frac{1}{d} & 0 \\ 0 & 1 \end{bmatrix} \left((cx_{i+1} + dx_{i+2}, u) + (-cx_{i+2}, u) \right) = \begin{bmatrix} \frac{1}{d} & 0 \\ 0 & 1 \end{bmatrix} (dx_{i+2}, u) = (x_{i+2}, u)$$

Now it is clear that we have 2-subspace which is fully analogue to the 2-

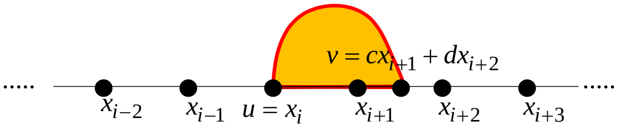
subspace which is generated as in sub case 8, that is equivalent with the sub case 2, which is fully described. So, the 2-vectors $(x_{i+1},x_{i+2}),(x_{i+2},x_i),(x_i,x_{i+1})$ all belong in M', where from we get that the kernel 2-subspace generated from them is also a subspace M'. That means that the kernel 2-subspace generated from $(x_{i+1},v),(v,u),(u,x_{i+1})$ is consisted also in the kernel 2-subspace generated from $(x_{i+1},x_{i+2}),(x_{i+2},x_i),(x_i,x_{i+1})$, and in M'. In any case, we have 2-subspace that is determined with $M'=M\cup L^2(x_i,x_{i+1},x_{i+2})$.

Sub case 7. $u=ax_i+bx_{i+1}$, $v=cx_{i+2}+dx_{i+3}$, where $ab\neq 0$ and $cd\neq 0$



It is clear that both vectors u and v are coordinates of some 2-vectors from the 2-vector space M (to be more clear see the drawing up for this sub case). The question is whether the vectors u and v are loops of two loop 2-subspaces of the new 2-subspace.

Sub case 8. $u=x_i$, $v=cx_{i+1}+dx_{i+2}$, $cd\neq 0$



It is clear that both vectors u and v are coordinates of some 2-vectors from the 2-vector space M (to be more clear see the drawing up for this sub case). The question is whether the vectors u and v are loops of two loop 2-subspaces of the new 2-subspace. Also, it is important to find the forms of the elements form the new 2-subspace M'. Now, since the 2-vectors $(v,u),(x_{i+1},u)\in M'$, we get that also the 2-vector

$$\begin{bmatrix}\frac{1}{d} & 0\\ 0 & 1\end{bmatrix}\left((v,u)+\begin{bmatrix}-c & 0\\ 0 & 1\end{bmatrix}(x_{i+1},u)\right)=\begin{bmatrix}\frac{1}{d} & 0\\ 0 & 1\end{bmatrix}\left((cx_{i+1}+dx_{i+2},x_i)+(-cx_{i+1},x_i)\right)=\begin{bmatrix}\frac{1}{d} & 0\\ 0 & 1\end{bmatrix}(dx_{i+2},x_i)=(x_{i+2},x_i)\in M'$$

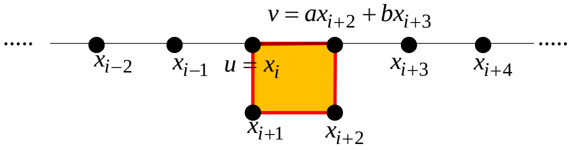
According to this, in this new 2-subspace belong the 2-vectors

$$(u,x_{i+1})=(x_i,x_{i+1}),(x_{i+1},x_{i+2}),(x_{i+2},x_i)=(x_{i+2},u),$$

and also the kernel subspace generated of the vectors x_i,x_{i+1},x_{i+2}. Now it is clear that this extension is equal to the extension from the sub case 2 from this case.

Sub case 9. $u=x_i$, $v=ax_{i+2}+bx_{i+3}$

In this sub case is clear that the 2-vectors $(u=x_i,x_{i+1}),(x_{i+1},x_{i+2}),(x_{i+2},v=ax_{i+2}+bx_{i+3})$ and $(v=ax_{i+2}+bx_{i+3},u)$ are four 2-vectors which form cyclic 2-subspace. The question what happens with the vector $v=ax_{i+2}+bx_{i+3}$ is implied here, i.e. whether this vector is a loop vector.



Sub case 10. $u = x_i$, $v = \alpha_j x_j + \alpha_{j+1} x_{j+1} + \alpha_{j+2} x_{j+2} + \alpha_{j+3} x_{j+3}$, $\alpha_j \alpha_{j+3} \neq 0$.

In this situation we have one vector which is a coordinate of a 2-vector from M . This is secured with the condition $\alpha_j \alpha_{j+3} \neq 0$. The reviews are the most common as in all other cases. But here the mutual ratio between i and j must be considered. Because of that we have more situations.

Situation 1. $i = j+1$(it is completely analogous and symmetrical $i = j+2$)

Now, we have situation in which $u = x_i$, $v = \alpha_{i-1} x_{i-1} + \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$, $\alpha_{i-1} \alpha_{i+2} \neq 0$.

In this situation the vector v plays the same role as in the sub case 1 from case 2. The reviews are completely analogous as in that sub case. Because $\alpha_{i-1} \alpha_{i+2} \neq 0$,i.e. $\alpha_{i-1}, \alpha_{i+2} \neq 0$ we have that the 2-vectors $(x_{i-1}, v), (x_i, v), (x_{i+1}, v), (x_{i+2}, v)$ are not neither from M neither from M' . From the other side, the 2-vectors $(x_{i-1}, u), (v, u), (x_{i+2}, u)$ are from M' , and also the branch 2-subspace determined with them is a 2-subspace from M' . According to this,

$$M' = M \cup \bigcup_{w \in L(x_{i-1}, v, x_{i+1})} L(w, u) = M \cup \bigcup_{w \in L(x_{i-1}, v, x_{i+1})} L(w, x_i)$$

Situation 2. $i = j$. Now we have a situation in which $u = x_i$, $v = \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2} + \alpha_{i+3} x_{i+3}$, $\alpha_i \alpha_{i+3} \neq 0$.

So, we have ordered pair $(u, v) = (x_i, \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2} + \alpha_{i+3} x_{i+3})$. Because (x_i, x_{i-1}) and (x_i, x_{i+1}) , we get that the vector x_i is a loop. But here, let us note that the 2-vector (v, x_{i+3}) is not a 2-vector from M' , because $\alpha_i \alpha_{i+3} \neq 0$, so also $\alpha_i, \alpha_{i+3} \neq 0$

In this case the extension is

$$M' = M \cup \bigcup_{w \in L(x_{i-1}, v, x_{i+1})} L(w, u) = M \cup \bigcup_{w \in L(x_{i-1}, v, x_{i+1})} L(w, x_i) .$$

Situation 3. $i = j-1$ Now we have a situation in which $u = x_i$, $v = \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2} + \alpha_{i+3} x_{i+3} + \alpha_{i+4} x_{i+4}$, $\alpha_{i+1} \alpha_{i+4} \neq 0$.

In this situation we have that $(x_{i+1}, v), (x_{i+2}, v), (x_{i+3}, v), (x_{i+4}, v)$ are 2-vectors which doesn't belong neither in M neither in M' (because $\alpha_{i+1} \alpha_{i+4} \neq 0$, i.e. $\alpha_{i+1}, \alpha_{i+4} \neq 0$). According to this, in this situation we have as in the other cases that the 2-vectors $(x_{i-1}, u), (v, u), (x_{i+1}, u)$ form loop 2-subspace which has the form

$$\bigcup_{w \in L(x_{i-1}, v, x_{i+1})} L(w, u) ,$$

and the 2-subspace in this case is $M' = M \cup \bigcup_{w \in L(x_{i-1}, v, x_{i+1})} L(w, u)$

Situation 4. $i = j-2$ Now we have a situation in which $u = x_i$, $v = \alpha_{i+2} x_{i+2} + \alpha_{i+3} x_{i+3} + \alpha_{i+4} x_{i+4} + \alpha_{i+5} x_{i+5}$, $\alpha_{i+2} \alpha_{i+5} \neq 0$

From the construction it is clear that $(v, x_{i+2}), (v, x_{i+3}), (v, x_{i+4}), (v, x_{i+5})$, as 2-vectors are not from M' , and also do not belong in M (that is from the condition that $\alpha_{i+2} \alpha_{i+5} \neq 0$, i.e. $\alpha_{i+2}, \alpha_{i+5} \neq 0$. According to this, in this situation the 2-vectors $(x_{i-1},u),(v,u),(x_{i+1},u)$ form a loop 2-subspace in the following form $\bigcup\limits_{w\in L(x_{i-1},v,x_{i+1})} L(w,u)$, and the new 2-subspace M' will be $M'=M\cup \bigcup\limits_{w\in L(x_{i-1},v,x_{i+1})} L(w,u)$.

Subcase11. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}+\alpha_{j+3}x_{j+3}$, where $\alpha_j\alpha_{j+3}\neq 0$ and $\alpha_i\alpha_{i+1}\neq 0$.

This case is possible because the element v is not a coordinate of none of the elements from the 2-subspace M, but it is an element of the vector space X and is a coordinate of the 2-vector (u,v). The same case can be considered also for a vector v, with the form $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}+\alpha_{j+3}x_{j+3}+...+\alpha_{j+\kappa}x_{j+\kappa}$, for any k which is greater than 3. There is essentially no difference. But here the mutual ratio between i and j must be considered. Because of that we have more situations.

Situation 1. $j=i-1$, $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\beta_{i-1}x_{i-1}+\beta_i x_i+\beta_{i+1}x_{i+1}+\beta_{i+2}x_{i+2}$, where $\beta_{i-1}\beta_{i+2}\neq 0$ and $\alpha_i\alpha_{i+1}\neq 0$.

Because $\beta_{i-1}\beta_{i+2}\neq 0$, i.e. $\beta_{i-1},\beta_{i+2}\neq 0$, we get that the vector v is not a coordinate of the 2-vector from M. According to this, the 2-vector (u,v) is completely the same as the sub case 2 from the case 2. So, in this case we will have 2-subspace determined with

$$M'=M\cup S_K,$$

where S_K is a 2-subspace which is a loop one with loop center u. This loop 2-subspace will be determined with $(x_i,u),(x_{i+1},u),(v,u)$. so, we have that

$$M'=M\cup \bigcup_{w\in L(x_i,x_{i+1},v)} L(w,u)\times L(w,u).$$

Situation 2. $j=i$, $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\beta_i x_i+\beta_{i+1}x_{i+1}+\beta_{i+2}x_{i+2}+\beta_{i+3}x_{i+3}$, where $\beta_i\beta_{i+3}\neq 0$ and $\alpha_i\alpha_{i+1}\neq 0$.

From the condition $\beta_i\beta_{i+3}\neq 0$, i.e. from $\beta_i,\beta_{i+3}\neq 0$, we get that the vector v is not a coordinate of any 2-vector from M. According to this, in this situation also we will have that the vector u will become a loop element and will generate the loop determined same as in the previous case with $(x_i,u),(x_{i+1},u),(v,u)$. So, we have that

$$M'=M\cup \bigcup_{w\in L(x_i,x_{i+1},v)} L(w,u)\times L(w,u).$$

Situation 3. $j=i+1$, $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\beta_{i+1}x_{i+1}+\beta_{i+2}x_{i+2}+\beta_{i+3}x_{i+3}+\beta_{i+4}x_{i+4}$, where $\beta_{i+1}\beta_{i+4}\neq 0$ and $\alpha_i\alpha_{i+1}\neq 0$.

From the condition of the case $\beta_{i+1}\beta_{i+4}\neq 0$, i.e. $\beta_{i+1},\beta_{i+4}\neq 0$ we have that the vector v, as in the previous situations of this sub case, is not a coordinate of a 2-vector from M. Now, it is clear that the 2-vectors $(x_i,u),(x_{i+1},u),(v,u)$ form a loop 2-subspace, and the whole 2-subspace M' now will have the following form

$$M'=M\cup \bigcup_{w\in L(x_i,x_{i+1},v)} L(w,u)\times L(w,u).$$

It is clear that the vector v as such can only be in a combination for a 2-vector. It cannot be obtained as a coordinate of vectors from M, even it is obtained from the generators of M.

Situation 4. $j=i+2$ $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\beta_{i+2}x_{i+2}+\beta_{i+3}x_{i+3}+\beta_{i+4}x_{i+4}+\beta_{i+5}x_{i+5}$, where $\beta_{i+2}\beta_{i+5}\neq 0$ and $\alpha_i\alpha_{i+1}\neq 0$.

This situation is completely the same as the previous.

Situation 5. $j=i+3$ $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\beta_{i+2}x_{i+2}+\beta_{i+3}x_{i+3}+\beta_{i+4}x_{i+4}+\beta_{i+5}x_{i+5}$, where $\beta_{i+2}\beta_{i+5}\neq 0$ and $\alpha_i\alpha_{i+1}\neq 0$.

This situation is completely the same as the previous.

Sub case 12. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}+\alpha_{i+3}x_{i+3}+\ldots+\alpha_{i+k}x_{i+k}$ and $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}+\alpha_{j+3}x_{j+3}+\ldots+\alpha_{j+s}x_{j+s}$, where $k,s\geq 3$. In this situation, neither the vector u nor the vector v are not coordinates of a 2-vector from M, so, according to this, this sub case is the same as the case 1. Between them is not possible to perform an operation. In other words, for any 2-vector (x,y) and the 2-vector (u,v) cannot be performed the operation addition of 2-vectors. In this case, automatically come to the situation completely analogous as the situation in the case 1.

Sub case 13. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}$, $\alpha_j\alpha_{j+2}\neq 0$ and $\alpha_i\alpha_{i+1}\neq 0$.

This case is possible, where u and v are vectors which are coordinates of some 2-vectors from the 2-subspace M. The condition $\alpha_j\alpha_{j+2}\neq 0$, i.e. $\alpha_j,\alpha_{j+2}\neq 0$ in general case ensures that the vector v cannot be from the form $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}$ or from the form $v=\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}$, which as a sub case of this case we consider in the sub case 15. Still, here it is ensured that it is at least from the form $v=\alpha_j x_j+\alpha_{j+2}x_{j+2}$ which will be considered.

Now, separately we will consider the addition of 2-vectors in this sub case.

Situation 1. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}$, where $\alpha_{i-1}\alpha_{i+1}\neq 0$.

Let's note that the vector $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}$ is on the branch $L^2(\alpha_{i-1}x_{i-1}+\alpha_{i+1}x_{i+1},x_i)$. But, as mentioned, it is not either in $\{A(x_{i-1},x_i)/A\in M_2(\Phi)\}$ nor in $\{B(x_i,x_{i+1})/B\in M_2(\Phi)\}$, and for this key condition is $\alpha_{i-1}\alpha_{i+1}\neq 0$, i.e. $\alpha_{i-1},\alpha_{i+1}\neq 0$. Here, let's note that the 2-vectors $(x_i,u),(u,v),(v,x_i)$ are 2-vectors such that two of them belong in the 2-subspace M and one of them, i.e. (u,v) does not belong in M. But, however, those three vectors form a 2-subspace $S_{u,v}$ which is a loop 2-subspace.

It is worth mentioning that αv (belongs in the one-dimensional vector space generated by v) belongs in M, i.e. belongs in $L^2(\alpha_{i-1}x_{i-1}+\alpha_{i+1}x_{i+1},x_i)$. But, here also the 2-vectors $(x_i,u),(u,\alpha v),(\alpha v,x_i)$ are also 2-vectors from M. Here, $(u,\alpha v)=A(u,v)=\begin{bmatrix}1&0\\0&\alpha\end{bmatrix}(u,v)$. According to this, this is a new loop 2-subspace which also belongs in M'.

Situation 2. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$, where $\alpha_i\alpha_{i+2}\neq 0$.

This situation is completely analogous to the previous situation of this sub case, just the vectors has exchanged its places. Here, the 2-vectors which build this 2-subspace are the 2-vectors $(u,x_{i+1}),(x_{i+1},v),(v,u)$. Of course it is a kernel 2-subspace.

Situation 3. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}+\alpha_{i+3}x_{i+3}$, where $\alpha_{i+1}\alpha_{i+3}\neq 0$

In this situation we have that $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$ is a vector that belongs to the vector subspace $\{A(x_i,x_{i+1})/A\in M_2(\Phi)\}$. On the other hand, the vector $v=\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}+\alpha_{i+3}x_{i+3}$ belongs in the branch 2-subspace determined with $L^2(\alpha_{i+1}x_{i+1}+\alpha_{i+3}x_{i+3},x_{i+2})$. According to this, the 2-vector (u,v) is not a vector from M, but it is a vector from M'. So, the vector v is a coordinate of 2-vectors from M. Because of the nature of the operations over the vectors from M', we have that also the vector αv has the same nature. But, here for the vector $(u,\alpha v)$ we have that it is also from M'. Indeed for $(u,\alpha v)=\begin{bmatrix}1 & 0\\ 0 & \alpha\end{bmatrix}(u,v)=A(u,v)\in M'$. Now, it is clear that the 2-vectors $(u,x_{i+1}),(x_{i+1},x_{i+2})(x_{i+2},\alpha v),(\alpha v,u)$ are also generators of one cyclic 2-subspace.

Situation 4. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\alpha_{i+2}x_{i+2}+\alpha_{i+3}x_{i+3}+\alpha_{i+4}x_{i+4}$, where $\alpha_{i+2}\alpha_{i+4}\neq 0$

In this situation, everything is the same as in the previous sub case, except that the number of generator elements of the cyclic 2-subspace is for one greater than before, which gets us to a different situation.

Sub case14. $u=x_i$, $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}$, $\alpha_j\alpha_{j+2}\neq 0$.

This case is possible, and u and v are vectors which are coordinates of some 2-vectors from the 2-subspace M. It is worth mentioning that the condition $\alpha_j\alpha_{j+2}\neq 0$, i.e. $\alpha_j,\alpha_{j+2}\neq 0$ is key condition, because we will not have a situation in which the vector v can be in the form $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}$ or in the form $v=\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}$, which as a situation we have as a sub case 8 or sub case 9 in this case. The least variant is the vector v to be in the form $v=\alpha_j x_j+\alpha_{j+2}x_{j+2}$. If we put that i is fixed index, and only j is variable, then we have the following situations:

Situation 1. $u=x_i$, $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $\alpha_{i-1}\alpha_{i+1}\neq 0$.

In this situation we have a 2-vector $(u,v)=(u=x_i,v)\in M\subseteq M'$. So, in this situation we do not have extension of the 2-subspace M.

Situation 2. $u=x_i$, $v=\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$. $\alpha_i\alpha_{i+2}\neq 0$.

In this situation we have the 2-vectors $(u,x_{i+1}),(x_{i+1},v),(v,u)$ which solely for themselves form loop 2-subspace, which we will denote with S. So, now we have extension in the form $M'=M\cup S$.

Situation 3. $u=x_i$, $v=\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}+\alpha_{i+3}x_{i+3}$, $\alpha_{i+1}\alpha_{i+3}\neq 0$.

It is clear that the 2-vectors $(u,x_{i+1}),(x_{i+1},x_{i+2}),(x_{i+2},v),(v,u)$ is a fourth vectors which form cyclic 2-subspace from X^2 and with that a cyclic 2-subspace from the

new 2-subspace M'. Now it is clear that the new 2-subspace which is a 2-subspace from X^2, has the following form $M' = M \cup S$,
where S is the cyclic 2-subspace which is previously described.

Situation 4. $u = x_i$, $v = \alpha_{i+2}x_{i+2} + \alpha_{i+3}x_{i+3} + \alpha_{i+4}x_{i+4}$, $\alpha_{i+2}\alpha_{i+4} \neq 0$.

In this situation we have totally analogous situation as before, just that the number of generator elements is for one greater from the previous situation.

Sub case 15. $u = \alpha_i x_i + \alpha_{i+1}x_{i+1}$, $v = \alpha_j x_j + \alpha_{j+1}x_{j+1}$. $\alpha_i\alpha_{i+1} \neq 0$, $\alpha_j\alpha_{j+1} \neq 0$.

In this sub case we have mutual relationship between i and j. In this sub case we will consider that i as an index is fixed, and only j will be variable. Here we have the following situations:

Situation 1. $u = \alpha_i x_i + \alpha_{i+1}x_{i+1}$, $v = \beta_{i-1}x_{i-1} + \beta_i x_i$.

In this situation we have a 2-vector (u,v) which has the form $(u,v) = (\alpha_i x_i + \alpha_{i+1}x_{i+1}, \beta_{i-1}x_{i-1} + \beta_i x_i)$. It is clear that the 2-vectors $(u, x_i) = (\alpha_i x_i + \alpha_{i+1}x_{i+1}, x_i)$ and $(x_i, v) = (x_i, \beta_{i-1}x_{i-1} + \beta_i x_i)$ are 2-vectors from the space from the beginning. According to that, we have a situation that these three 2-vectors make a new kernel 2-subspace from M'. This 2-subspace is as the 2-subspace from the sub case 6 of this paper. So, in this case we have a 2-subspace which is a kernel one, i.e. the whole extension is like in the sub case 6 from this paper, i.e. $M' = M \cup L^2(x_i, x_{i+1}, x_{i+2})$

Situation 2. $u = \alpha_i x_i + \alpha_{i+1}x_{i+1}$, $v = \beta_i x_i + \beta_{i+1}x_{i+1}$

In this situation we have a 2-vector (u,v) which is in the 2-subspace M, i.e. it can be written as a 2-vector in the form $A(x_i, y_i)$, where $A = \begin{bmatrix} \alpha_i & \alpha_{i+1} \\ \beta_i & \beta_{i+1} \end{bmatrix}$, i.e.

$$(u,v) = A(x_i, x_{i+1}) = \begin{bmatrix} \alpha_i & \alpha_{i+1} \\ \beta_i & \beta_{i+1} \end{bmatrix}(x_i, x_{i+1}) = (\alpha_i x_i + \alpha_{i+1}x_{i+1}, \beta_i x_i + \beta_{i+1}x_{i+1})$$

So, in this situation we do not have any extension.

Situation 3. $u = \alpha_i x_i + \alpha_{i+1}x_{i+1}$, $v = \beta_{i+1}x_{i+1} + \beta_{i+2}x_{i+2}$

In this situation we have a 2-subspace which is totally the same as in the previous situation 1.

Situation 4. $u = \alpha_i x_i + \alpha_{i+1}x_{i+1}$, $v = \alpha_{i+2}x_{i+2} + \alpha_{i+3}x_{i+3}$

In this situation we have a 2-subspace which is the same as the sub case 7 from this case and here we will not describe it.

Situation 5. $u = \alpha_i x_i + \alpha_{i+1}x_{i+1}$, $v = \alpha_{i+3}x_{i+3} + \alpha_{i+4}x_{i+4}$

In this situation 5 we have a 2-subspace which is generated from $u = \alpha_i x_i + \alpha_{i+1}x_{i+1}$ and $v = \alpha_{i+3}x_{i+3} + \alpha_{i+4}x_{i+4}$ which is equal to subcase 7.

Sub case 16. $u = \alpha_i x_i + \alpha_{i+1}x_{i+1} + \alpha_{i+2}x_{i+2}$, $v = \beta_j x_j + \beta_{j+1}x_{j+1} + \beta_{j+2}x_{j+2}$, $\alpha_i\alpha_{i+2} \neq 0$, $\alpha_j\alpha_{j+2} \neq 0$

From the conditions $\alpha_i\alpha_{i+2} \neq 0$ and $\alpha_j\alpha_{j+2} \neq 0$ it is clear that these vectors cannot be in the form $u = \alpha_i x_i + \alpha_{i+1}x_{i+1}$ or $u = \alpha_{i+1}x_{i+1} + \alpha_{i+2}x_{i+2}$, i.e. $v = \beta_{j+1}x_{j+1} + \beta_{j+2}x_{j+2}$

or $v = \beta_j x_j + \beta_{j+1} x_{j+1}$. That case is considered in the sub case 15 of this case 3. Also, it is clear that the vector $v = \beta_j x_j + \beta_{j+1} x_{j+1} + \beta_{j+2} x_{j+2}$ belongs in the 2-subspace

$L(\beta_j x_j + \beta_{j+2} x_{j+2}, x_{j+1}) \times L(\beta_j x_j + \beta_{j+2} x_{j+2}, x_{j+1})$,

and also that the vector $u = \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$ belongs in the 2-subspace

$L(\alpha_i x_i + \alpha_{i+2} x_{i+2}, x_{i+1}) \times L(\alpha_i x_i + \alpha_{i+2} x_{i+2}, x_{i+1})$.

Here, it is clear that the 2-vectors (v, x_{j+1}) and (u, x_{i+1}) also belong in the 2-subspaces

$L(\beta_j x_j + \beta_{j+2} x_{j+2}, x_{j+1}) \times L(\beta_j x_j + \beta_{j+2} x_{j+2}, x_{j+1})$ and

$L(\alpha_i x_i + \alpha_{i+2} x_{i+2}, x_{i+1}) \times L(\alpha_i x_i + \alpha_{i+2} x_{i+2}, x_{i+1})$ accordingly.

But, in this sub case we have a mutual relationship between i and j, which we will thoroughly consider.

Situation 1. $u = \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$, $v = \alpha_{i-1} x_{i-1} + \alpha_i x_i + \alpha_{i+1} x_{i+1}$

From the definition of M' certainly (u, v) is a 2-vector which belong in it. But now we have that the four 2-vectors $(v, x_i), (x_i, x_{i+1}), (x_{i+1}, u), (u, v)$ belong in M', so the cyclic 2-subspace generated with them, also belongs.

Situation 2. $u = \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$, $v = \beta_i x_i + \beta_{i+1} x_{i+1} + \beta_{i+2} x_{i+2}$

In this situation we have that the 2-vectors $(v, x_{i+1}), (x_{i+1}, u), (u, v)$ are three 2-vectors which belong in M', so, according to that in the same 2-subspace will belong also the kernel 2-subspace generated by them. So, we will have that $L^2(u, v, x_{i+1}) \subset M'$, and now it is clear that $M' = M \cup L^2(u, v, x_{i+1})$.

Situation 3. $u = \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$, $v = \beta_{i+1} x_{i+1} + \beta_{i+2} x_{i+2} + \beta_{i+3} x_{i+3}$

It is obvious that this 2-subspace from this situation is totally analogous to the 2-subspace from the situation 1 from this sub case.

Situation 4. $u = \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$, $v = \beta_{i+2} x_{i+2} + \beta_{i+3} x_{i+3} + \beta_{i+4} x_{i+4}$

In this situation we have that the five 2-vectors

$(u, x_{i+1}), (x_{i+1}, x_{i+2}), (x_{i+2}, x_{i+3}), (x_{i+3}, v), (v, u)$,

form a cyclic 2-subspace which at the same time is a 2-subspace from M', too. We will denote it with S_K. So $M' = M \cup S_K$

Situation 5. $u = \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$, $v = \beta_{i+3} x_{i+3} + \beta_{i+4} x_{i+4} + \beta_{i+5} x_{i+5}$.

According to the previous notes, from the introduction of these situations, as well as from the general sub case, we have that the 2-vectors

$(u, x_{i+1}), (x_{i+1}, x_{i+2}), (x_{i+2}, x_{i+3}), (x_{i+3}, x_{i+4})(x_{i+4}, v), (v, u)$

form a cyclic 2-subspace which is a 2-subspace generated from six 2-vectors. This 2-subspace is also a 2-subspace from M'. We will denote it with S_K. So,

$M' = M \cup S_K$.

3. EXTENSION OF A TWO-SKEW-SYMMETRIC LINEAR FORM

In this part we will consider that the field Φ is the field of real numbers, $\mathbb{R}$.

Case 2, sub case 1

Theorem. *Let* $\Lambda : M \to \mathbb{R}$ *be a* 2*-skew-symmetric form such that* $\Lambda(x,y) \le p(x,y)$ *for every* $(x,y) \in M$, $p : X^2 \to \mathbb{R}$ *be a* 2*-semi norm and* M *is a branch* 2*-subspace of the* 2*-space* X^2. *Let* M' *be an extension of* M *as in sub case 1 of case 2. Then there exists a* 2*-skew-symmetric linear form* $\Lambda' : M' \to \mathbb{R}$ *such that*

$$\Lambda'/M = \Lambda$$

$$-p(-x,y) \le \Lambda'(x,y) \le p(x,y). \qquad (*)$$

Proof. We will choose two arbitrary elements from the 2-subspace M, which in the same time belong in the loop u. let that be the elements $(\alpha_1 x_{i-1} + \alpha x_{i+1}, u)$ and $(\alpha'_{i-1} x_{i-1} + \alpha x'_{i+1}, u)$. For the 2-skew-symmetric form Λ, according to the conditions of the theorem, we have that

$$\Lambda(\alpha_{i-1}x_{i-1} + \alpha x_{i+1}, u) + \Lambda(\alpha'_{i-1}x_{i-1} + \alpha x'_{i+1}, u) = \Lambda(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1} + \alpha'_{i-1}x_{i-1} + \alpha x'_{i+1}, u) \le$$

$$\le p(\alpha_{i-1}x_{i-1} + \alpha x_{i+1} + \alpha'_{i-1}x_{i-1} + \alpha_{i+1}x'_{i+1}, u) = p(\alpha_{i-1}x_{i-1} + \alpha x_{i+1} - v + \alpha'_{i-1}x_{i-1} + \alpha x'_{i+1} + v, u) \le$$

$$\le p(\alpha_{i-1}x_{i-1} + \alpha x_{i+1} - v, u) + + P(\alpha'_{i-1}x_{i-1} + \alpha x'_{i+1} + v, u)$$

In other words, the inequality is fulfilled

$$\Lambda(\alpha_{i-1}x_{i-1} + \alpha x_{i+1}, u) - p(\alpha_{i-1}x_{i-1} + \alpha x_{i+1} - v, u) \le p(\alpha'_{i-1}x_{i-1} + \alpha x'_{i+1} + v, u) - \Lambda(\alpha'_{i-1}x_{i-1} + \alpha x'_{i+1}, u)$$

Since $\alpha_{i-1}, \alpha_{i+1} \in \mathbb{R}$ and $\alpha'_{i-1}, \alpha'_{i+1} \in \mathbb{R}$ are arbitrary, we get that

$$\sup_{\alpha_{i-1},\alpha_{i+1}} \Lambda(\alpha_{i-1}x_{i-1} + \alpha x_{i+1}, u) - p(\alpha_{i-1}x_{i-1} + \alpha x_{i+1} - v, u) = d \le p(\alpha'_{i-1}x_{i-1} + \alpha x'_{i+1} + v, u) - \Lambda(\alpha'_{i-1}x_{i-1} + \alpha x'_{i+1}, u)$$

So, for arbitrary $\alpha_{i-1}, \alpha_{i+1}, \alpha'_{i-1}, \alpha'_{i+1} \in \mathbb{R}$, the inequalities are fulfilled

$$\Lambda(\alpha_1 x_{i-1} + \alpha x_{i+1}, u) - p(\alpha_1 x_{i-1} + \alpha x_{i+1} - v, u) \le d$$

$$d \le p(\alpha'_{i-1}x_{i-1} + \alpha'_{i+1}x_{i+1} + v, u) - \Lambda(\alpha'_{i-1}x_{i-1} + \alpha'_{i+1}x'_{i+1}, u)$$

i.e. $\Lambda(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}, u) - d \le p(\alpha_1 x_{i-1} + \alpha x_{i+1} - v, u) \qquad (1)$

$$\Lambda(\alpha'_{i-1}x_{i-1} + \alpha x'_{i+1}, u) + d \le p(\alpha'_{i-1}x_{i-1} + \alpha x'_{i+1} + v, u) \qquad (2)$$

Now, we will determine $\Lambda' : M' \to \mathbb{R}$ with

$$\Lambda'[A(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1} + \gamma v, u)] = (\det A)[\Lambda(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}, u) + \gamma d],\ \gamma \in \mathbb{R},$$

$$\Lambda'(x,y)] = \Lambda(x,y),\ (x,y) \in M.$$

According to this $\Lambda'/M = \Lambda$.

From the other side, if in instead of α_{i-1} and α_{i+1} we choose $\frac{\alpha_{i-1}}{t}$ and $\frac{\alpha_{i+1}}{t}$, $t > 0$ and if we use the properties of Λ and p accordingly, we get that

$$\Lambda(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}, u) - td \le p(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1} - tv, u). \qquad (3)$$

Fully analogous, if in (2) instead α'_{i-1} and α'_{i+1} we choose $\frac{\alpha'_{i-1}}{t}$ and $\frac{\alpha'_{i+1}}{t}$, $t > 0$ accordingly, and again, if we use the properties of Λ and p, we get that

$$\Lambda(\alpha'_{i-1}x_{i-1}+\alpha'_{i+1}x_{i+1},u)+td \le p(\alpha_{i-1}x_{i-1}+\alpha_{i+1}x_{i+1}+tv,u)\ . \qquad (4)$$

Now, from (3) and (4) we see that

$$\Lambda'(\alpha_{i-1}x_{i-1}+\alpha_{i+1}x_{i+1}+\gamma v,u)\le p(\alpha_{i-1}x_{i-1}+\alpha_{i+1}x_{i+1}+\gamma v,u)\ ,$$

where from it is clear that in general case $\Lambda'\le p$ on M'. in other words, the inequality (*) is fulfilled.

Case 2, sub case 2

Theorem. *Let* $\Lambda: M\to\mathbb{R}$ *be a* 2*-skew-symmetric form such that* $\Lambda(x,y)\le p(x,y)$ *for every* $(x,y)\in M$, $p: X^2\to\mathbb{R}$ *be a* 2*-semi norm and* M *is a branch* 2*-subspace of the* 2*-space* X^2. *Let* M' *be an extension of* M *as in sub case 2 of case 2. Then there exists a* 2*-skew-symmetric linear form* $\Lambda': M'\to\mathbb{R}$ *such that*

$$\Lambda'/M=\Lambda$$

$$-p(-x,y)\le\Lambda'(x,y)\le p(x,y)\ . \qquad (*)$$

Proof. We will choose two arbitrary elements from the 2-subspace M, which at the same time belong in the loop u. Let us note here that the choosing of the elements from this 2-subspace can be done in the following way

$$\begin{bmatrix}\alpha_i & \alpha_{i+1}\\ a & b\end{bmatrix}(x_i,x_{i+1})=(\alpha_i x_i+\alpha_{i+1}x_{i+1},u)\ ,$$

$$\begin{bmatrix}\beta_i & \beta_{i+1}\\ a & b\end{bmatrix}(x_i,x_{i+1})=(\beta_i x_i+\beta_{i+1}x_{i+1},u)\ ,$$

where $\det A\ne 0$, $A=\begin{bmatrix}\alpha_i & \alpha_{i+1}\\ \beta_i & \beta_{i+1}\end{bmatrix}$ and $\det\left(\begin{bmatrix}\alpha_i & \alpha_{i+1}\\ a & b\end{bmatrix}\right)$, $\det\left(\begin{bmatrix}\beta_i & \beta_{i+1}\\ a & b\end{bmatrix}\right)\ne 0$. In that case,

$$\begin{aligned}\Lambda(\alpha_i x_i+\alpha_{i+1}x_{i+1},u)+\Lambda(\beta_i x_i+\beta_{i+1}x_{i+1},u)&=\Lambda(\alpha_i x_i+\alpha_{i+1}x_{i+1}+\beta_i x_i+\beta_{i+1}x_{i+1},u)\le\\ &\le p(\alpha_i x_i+\alpha_{i+1}x_{i+1}+\beta_i x_i+\beta_{i+1}x_{i+1},u)=p(\alpha_i x_i+\alpha_{i+1}x_{i+1}-v+\beta_i x_i+\beta_{i+1}x_{i+1}+v,u)\le\\ &\le p(\alpha_i x_i+\alpha_{i+1}x_{i+1}-v,u)+p(\beta_i x_i+\beta_{i+1}x_{i+1}+v,u)\end{aligned}$$

In other words, the inequality holds

$$\Lambda(\alpha_i x_i+\alpha_{i+1}x_{i+1},u)-p(\alpha_i x_i+\alpha_{i+1}x_{i+1}-v,u)\le p(\beta_i x_i+\beta_{i+1}x_{i+1}+v,u)-\Lambda(\beta_i x_i+\beta_{i+1}x_{i+1},u)\ .$$

Now, from the arbitrariness of $\alpha_i,\alpha_{i+1}\in\mathbb{R}$ and of $\beta_i,\beta_{i+1}\in\mathbb{R}$ we have that

$$\sup_{\alpha_i,\alpha_{i+1}\in\mathbb{R}}[\Lambda(\alpha_i x_i+\alpha_{i+1}x_{i+1},u)-p(\alpha_i x_i+\alpha_{i+1}x_{i+1}-v,u)]=d\le p(\beta_i x_i+\beta_{i+1}x_{i+1}+v,u)-\Lambda(\beta_i x_i+\beta_{i+1}x_{i+1},u)$$

So, for arbitrary $\alpha_i,\alpha_{i+1}\in\mathbb{R}$ and $\beta_i,\beta_{i+1}\in\mathbb{R}$ the inequalities hold

$$\Lambda(\alpha_i x_i+\alpha_{i+1}x_{i+1},u)-p(\alpha_i x_i+\alpha_{i+1}x_{i+1}-v,u)\le d\ - \qquad (1)$$

$$d\le p(\beta_i x_i+\beta_{i+1}x_{i+1}+v,u)-\Lambda(\beta_i x_i+\beta_{i+1}x_{i+1},u) \qquad (2)$$

Now, let $\Lambda': M'\to\mathbb{R}$ be determined with

$$\Lambda'[A(\alpha_i x_i+\alpha_{i+1}x_{i+1}+\gamma v,u)]=(\det A)[\Lambda(\alpha_i x_i+\alpha_{i+1}x_{i+1},u)+\gamma d]$$

$$\Lambda'(x,y)=\Lambda(x,y)\ ,\ (x,y)\in M$$

Here $\Lambda'/M=\Lambda$.

Let's substitute $\frac{\alpha_i}{t}$ and $\frac{\alpha_{i+1}}{t}$ instead α_i and α_{i+1} in the inequality (1), and in the inequality (2) we substitute $\frac{\beta_i}{t}$ and $\frac{\beta_{i+1}}{t}$ instead β_i and β_{i+1}. Then

$$\Lambda(\alpha_i x_i + \alpha_{i+1}x_{i+1}, u) - td \le p(\alpha_i x_i + \alpha_{i+1}x_{i+1} - tv, u) \qquad (3)$$

$$\Lambda(\beta_i x_i + \beta_{i+1}x_{i+1}, u) + td \le p(\beta_i x_i + \beta_{i+1}x_{i+1} + v, u) \qquad (4)$$

Now, from (3) and (4) it is clear that

$$\Lambda'(\alpha_i x_i + \alpha_{i+1}x_{i+1} + \gamma v, u) \le p(\alpha_i x_i + \alpha_{i+1}x_{i+1} + \gamma v, u)\ .$$

With this, the proof that $\Lambda' \le p$ through M', is completes, i.e. that (*) holds.

Case 2, sub case 3

There is no case, and there is no Hahn-Banach theorem.

Case 2, sub case 4

Theorem. *Let* $\Lambda : M \to \mathbb{R}$ *be a* 2*-skew-symmetric form such that* $\Lambda(x,y) \le p(x,y)$ *for every* $(x,y) \in M$, $p : X^2 \to \mathbb{R}$ *be a* 2*-semi norm and* M *is a two-sided branch* 2*-subspace of the* 2*-space* X^2. *Let* M' *be an extension of* M *as in sub case 4 of case 2. Then there exists a* 2*-skew-symmetric linear form* $\Lambda' : M' \to \mathbb{R}$ *such that*

$$\Lambda'/\,M = \Lambda$$

$$-p(-x,y) \le \Lambda'(x,y) \le p(x,y)\ . \qquad (*)$$

Proof. Let the vector u is given with $u = \alpha_i x_i + \alpha_{i+1}x_{i+1} + \alpha_{i+2}x_{i+2}$ (we will consider especially the case when $\alpha_i \alpha_{i+2} \ne 0$ - the rest of the situations are already considered). It is clear that we can choose 2-vectors in the form (u,x) and (u,y) which belong in the 2-vector subspace M. Indeed, we choose 2-vectors $(\alpha_i x_i, x_{i+1}), (\alpha_{i+2}x_{i+2}, x_{i+1})$ which according to the definition of M belong in M. But then, in M belongs also the 2-vector $(\alpha_i x_i, x_{i+1}) + (\alpha_{i+2}x_{i+2}, x_{i+1}) = (\alpha_i x_i + \alpha_{i+2}x_{i+2}, x_{i+1})$. We choose matrices $A, B \in M_2(\mathbb{R})$ given with $A = \begin{bmatrix} 1 & \alpha_{i+1} \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & \alpha_{i+1} \\ 0 & \alpha_{i+1} \end{bmatrix}$, and we get the 2-vectors

$(u, \alpha_i x_i + \alpha_{i+1}x_{i+2})$ and $(u, \alpha_{i+1}x_{i+1})$,

which belong in M. Now, it is clear that in this 2-subspace, belong every 2-vector in the form

$$C(u, \beta_i x_i + \beta_{i+1}x_{i+2}) = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} (u, \beta_i x_i + \beta_{i+1}x_{i+2}) = (u, \alpha(\beta_i x_i + \beta_{i+1}x_{i+2}))$$

as well as every 2-vector in the form

$$D(u, \beta_{i+1}x_{i+1}) = \begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix} (u, \beta_{i+1}x_{i+1}) = (u, \beta\beta_{i+1}x_{i+1})\cdot$$

Finally, in this 2-subspace M belongs also every 2-vector in the form

$$(u, \alpha(\beta_i x_i + \beta_{i+1}x_{i+2})) + (u, \beta\beta_{i+1}x_{i+1}) = (u, \alpha(\beta_i x_i + \beta_{i+1}x_{i+2}) + \beta\beta_{i+1}x_{i+1})\ ,$$

which can be obtained also in another way.

Let us now have two vectors in that form, which belong in the 2-subspace M and let it be the 2-vectors

$(u,\alpha'(\beta_i x_i+\beta_{i+1}x_{i+2})+\beta'\beta_{i+1}x_{i+1})$

$(u,\alpha(\beta_i x_i+\beta_{i+1}x_{i+2})+\beta\beta_{i+1}x_{i+1})$.

Now, the proof continues the same as in the previous two theorems like this one.

CONFLICT OF INTEREST

No conflict of interest was declared from the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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