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THREE THEOREMS ABOUT FIXED POINT FOR CONTRACTIONS IN A COMPLETE METRIC SPACE

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Abstract. In this paper are presented some generalizations of the R. Kannan, S. K. Chatterjea and P.V. Koparde and B.B. Waghmode theorems about common fixed points in a complete metric space (X,d). In doing so, we defined a continuous, injection and subsequentially convergent mapping T, and a function f. The function belongs to class Θ continuous monotony non-decreasing functions $f:[0,+\infty)\to[0,+\infty)$ such that $f^{-1}(0)=\{0\}$. In some results f is additionally defined as sub additive.

1. INTRODUCTION

The Banach principle for fixed point is well known in the literature. That is:

Let (X,d) be a metric space. The mapping $S:X\to X$ is said to be a contraction if there exists $\lambda\in(0,1)$ such that for all $x,y\in X$ holds that

$$d(Sx,Sy)\le\lambda d(x,y)\,. \tag{1}$$

If the metric space (X,d) is a complete metric space, then the mapping T for the condition (1) is satisfied has a unique fixed point.

R. Kannan, 1968 ([4]) generalized the Banach principle about a fixed point, as the following:

Theorem 1. If the mapping $S:X\to X$ for a complete metric space (X,d), satisfies the inequality

$$d(Sx,Sy)\le\lambda(d(x,Sx)+d(y,Sy))\,, \tag{2}$$

for $\lambda\in(0,\frac{1}{2})$ and $x,y\in X$, then S has a unique fixed point. □

If S satisfies the condition (2), then S is said to be Kannan type mapping.

S. K. Chatterjea, 1972 ([7]), defined similar conditions for contraction as the following:

Theorem 2. If the mapping $S:X\to X$ for a complete metric space (X,d) satisfies the inequality

$$d(Sx,Sy)\le\lambda(d(x,Sy)+d(y,Sx))\,, \tag{2}$$

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for $\lambda \in (0, \frac{1}{2})$ and $x, y \in X$, then S has a unique fixed point. □

If S satisfies the condition (2), then S is said to be Chatterjea type of mapping.

P. V. Koparde and B. B. Waghmode, 1991 ([3]), presented a new generalization of the Banach principle for a fixed point as the following:

Theorem 3. If the mapping $S: X \to X$ for a complete metric space (X, d) satisfies the inequality

$$d^2(Sx, Sy) \le \lambda(d^2(x, Sx) + d^2(y, Sy)), \tag{3}$$

for $\lambda \in (0, \frac{1}{2})$ and $x, y \in X$, then S has a unique fixed point. □

If S satisfies the condition (3), then S is Koparde-Waghmode type of mapping.

S. Moradi and D. Alimohammadi [9] generalized the R. Kannan result, using the sequentially convergent mappings. Some generalizations of The Kannan, Chatterjea and Koparde-Waghmode theorems are proven in [1], by using the sequentially convergent mappings, defined as the following:

Definition 1 ([8]). Let (X, d) be a metric space. A mapping $T: X \to X$ is said sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ also is convergence. A mapping T is said sub-sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ has a convergent subsequence.

S. Moradi и A. Beiranvand, [8] introduce the concept for T_f contractive mapping, by using Θ class of continuous monotony non-decreasing functions $f: [0, +\infty) \to [0, +\infty)$ such that $f^{-1}(0) = \{0\}$, defined as the following.

Definition 2 ([8]). Let (X, d) be a metric space, $S, T: X \to X$ and $f \in \Theta$. A mapping S is said T_f −contraction if there exist $\lambda \in (0,1)$ such that

$$f(d(TSx, TSy)) \le \lambda f(d(Tx, Ty)),$$

for all $x, y \in X$.

We must notice that, if $f \in \Theta$, then $f^{-1}(0) = \{0\}$ implies that $f(t) > 0$, for all $t > 0$. S. Moradi and A. Beiranvand proved that if S is T_f contractive mapping, then S has a unique fixed point. Then, M. Kir and H. Kiziltunc, [2] generalized the S. Moradi and A. Beiranvand result, for Kannan and Chatterjea type of mapping.

In our further consideration we will generalize the Kir and Kiziltunc results and will elaborate its application to the Koparde-Waghmoden type of mapping.

2. MAINS RESULTS

Theorem 4. Let (X,d) be a complete metric space $S:X\to X$, $f\in\Theta$ and the mapping $T:X\to X$ be continuous, injection and subsequentially convergent. If there exist $\alpha>0,\beta\ge 0$ such that $\alpha+2\beta\in(0,1)$ and

$$f(d(TSx,TSy))\le(\alpha+\beta)f(d(Tx,TSx))+\beta f(d(Ty,TSy)) \tag{4}$$

for all $x,y\in X$, then S has a unique fixed point.

Proof. Let x_0 be any point on X and let the sequence $\{x_n\}$ be defined as $x_{n+1}=Sx_n$, $n=0,1,2,3,\dots$. For $\lambda=\frac{\alpha+2\beta}{2-(\alpha+2\beta)}$ and since $\alpha+2\beta\in(0,1)$, $\alpha,\beta\ge 0$, we get that $\lambda\in(0,1)$. The inequality (4) implies

$$\begin{aligned} f(d(Tx_{n+1},Tx_n)) &= f(d(TSx_n,TSx_{n-1}))\\ &\le(\alpha+\beta)f(d(Tx_{n-1},TSx_{n-1}))+\beta f(d(Tx_n,TSx_n))\\ &=(\alpha+\beta)f(d(Tx_{n-1},Tx_n))+\beta f(d(Tx_n,Tx_{n+1})). \end{aligned}$$

Analogously,

$$f(d(Tx_{n+1},Tx_n))\le\beta f(d(Tx_{n-1},Tx_n))+(\alpha+\beta)f(d(Tx_n,Tx_{n+1})).$$

By adding the last two inequalities , we get the following

$$f(d(Tx_{n+1},Tx_n))\le\lambda f(d(Tx_n,Tx_{n-1})), \tag{5}$$

for each $n=1,2,3,\dots$. The inequality (5) implies that

$$f(d(Tx_{n+1},Tx_n))\le\lambda^n f(d(Tx_1,Tx_0)), \tag{6}$$

for each $n=1,2,3,\dots$. Further, the inequalities (4) and (6) imply that for all $m,n\in\mathbb{N}$ $n>m$

$$\begin{aligned} f(d(Tx_n,Tx_m)) &= f(d(TSx_{n-1},TSx_{m-1}))\\ &\le(\alpha+\beta)f(d(TSx_{n-1},Tx_{n-1}))+\beta f(d(TSx_{m-1},Tx_{m-1}))\\ &=(\alpha+\beta)f(d(Tx_n,Tx_{n-1}))+\beta f(d(Tx_m,Tx_{m-1}))\\ &\le[(\alpha+\beta)\lambda^{n-1}+\beta\lambda^{m-1}]f(d(Tx_1,Tx_0)) \end{aligned}$$

holds true. Analogously,

$$f(d(Tx_n,Tx_m))\le[(\alpha+\beta)\lambda^{m-1}+\beta\lambda^{n-1}]f(d(Tx_1,Tx_0))].$$

By adding the last two inequalities, we get that

$$f(d(Tx_n,Tx_m))\le\frac{\alpha+2\beta}{2}(\lambda^{m-1}+\lambda^{n-1})f(d(Tx_1,Tx_0))].$$

The last inequality implies that

$$\lim_{m,n\to\infty} f(d(Tx_n,Tx_m))=0,$$

and since $f\in\Theta$ we get that $\lim\limits_{m,n\to\infty} d(Tx_n,Tx_m)=0$. Therefore, the sequence $\{Tx_n\}$ is Caushy sequence. But, X is complete metric space, and therefore the sequence $\{Tx_n\}$ is a convergent sequence. The mapping $T:X\to X$ is subsequentially convergent, therefore the sequence $\{x_n\}$ consists a convergent subsequence $\{x_{n(k)}\}$, i.e. it exists $u\in X$ so that $\lim\limits_{k\to\infty} x_{n(k)}=u$. The continuity of T implies that $\lim\limits_{k\to\infty} Tx_{n(k)}=Tu$. Further, $\{Tx_{n(k)}\}$ is a subsequence of the convergent sequence $\{Tx_n\}$, therefore $\lim\limits_{n\to\infty} Tx_n=\lim\limits_{k\to\infty} Tx_{n(k)}=Tu$.

It will be proven that $u\in X$ is fixed point for the mapping S. Now,

$$f(d(TSu,Tx_{n+1}))=f(d(TSu,TSx_n))\le(\alpha+\beta)f(d(TSu,Tu))+\beta f(d(TSx_n,Tx_n))$$
$$=(\alpha+\beta)f(d(TSu,Tu))+\beta f(d(Tx_{n+1},Tx_n))$$

holds true. Analogously,

$$f(d(TSu,Tx_{n+1}))\le\beta f(d(TSu,Tu))+(\alpha+\beta)f(d(Tx_{n+1},Tx_n)),$$

therefore

$$f(d(TSu,Tx_{n+1}))\le\tfrac{\alpha+2\beta}{2}[f(d(TSu,Tu))+f(d(Tx_{n+1},Tx_n))]$$

For $n\to\infty$, in the inequality above, $\lim\limits_{n\to\infty} Tx_n=Tu$ and the properties of f and the metrics imply that

$$f(d(TSu,Tu))\le\tfrac{\alpha+2\beta}{2}[f(d(TSu,Tu))+f(0)]$$

holds true. But $1-\frac{\alpha+2\beta}{2}>0$ and $f^{-1}(0)=\{0\}$. Therefore, the above inequality implies that $d(TSu,Tu)=0$, i.e. $TSu=Tu$. Finally, T is injection, and therefore $Su=u$, that is the mapping S has a fixed point.

Let $u,v\in X$ be two fixed points for S, i.e. $Su=u$ and $Sv=v$. So, (4) implies that

$$f(d(Tu,Tv))=f(d(TSu,TSv))\le(\alpha+\beta)[f(d(Tu,TSu))+\beta f(d(Tv,TSv))]=0,$$

holds true, that is $d(Tu,Tv)=0$. Therefore, $Tu=Tv$. But, T is injection, and therefore $u=v$, that is T has a unique fixed point. ■

Corollary 1. Let (X,d) be a complete metric space, $S:X\to X$ and $f\in\Theta$. If there exist $\alpha,\beta\ge0$ such that $\alpha+2\beta\in(0,1)$ and

$$f(d(Sx,Sy))\le(\alpha+\beta)f(d(x,Sx))+\beta f(d(y,Sy)),$$

for all $x,y\in X$, then S has a unique fixed point and for each $x_0\in X$ the sequence $\{S^n x_0\}$ converges to the fixed point.

Proof. The mapping $Tx = x$, for each $x \in X$ is continuous, injection and sequentially convergent. Therefore, the corollary is directly implied by Theorem 4 for $Tx = x$. ■

Corollary 2. Let (X,d) be a complete metric space, $S: X \to X$ and the mapping $T: X \to X$ is continuous, injection and subsequentially convergent. If there exist $\alpha,\beta \geq 0$ such that $\alpha + 2\beta \in (0,1)$ and

$$d(TSx,TSy) \leq (\alpha+\beta)d(Tx,TSx) + \beta d(Ty,TSy)$$

for all $x, y \in X$, then S has a unique fixed point.

Proof. The function $f(t) = t$, $t \geq 0$ is monotony increasing and $f^{-1}(0) = \{0\}$. Therefore, the corollary is a direct implication of Theorem 4 for $f(t) = t$. ■

Comment 1. 1) For $\alpha = 0$ и $\beta = \lambda$, the Theorem 4 is transformed as the Theorem2.1 [2].

2) If we take into a consideration that the mapping $Tx = x$, for all $x \in X$ is continuous, injection and subsequentially convergent, the Corollary 2 implies that if for the mapping $S: X \to X$ exist $\alpha,\beta \geq 0$ such that $\alpha + 2\beta \in (0,1)$ and

$$d(Sx,Sy) \leq (\alpha+\beta)d(x,Sx) + \beta d(y,Sy), \tag{7}$$

for all $x, y \in X$, then S has a unique fixed point.

3) For $\alpha = 0$ and $\beta = \lambda$ in (7), we get that the Theorem 4 implies the Theorem 1.

Theorem 5. Let (X,d) be a complete metric space $S: X \to X$, $f \in \Theta$ and the mapping $T: X \to X$ be continuous, injection and subsequentially convergent. If it exist $\alpha > 0, \beta \geq 0$ such that $\alpha + 2\beta \in (0,1)$ and

$$f(d^2(TSx,TSy)) \leq (\alpha+\beta) f(d^2(Tx,TSx)) + \beta f(d^2(Ty,TSy))$$

for all $x, y \in X$, then S has a unique fixed point.

Proof. It is obvious that $g(t) = t^2, t \geq 0$ is a function of the Θ class. Further, if $f, g \in \Theta$, then $f \circ g \in \Theta$, therefore the truth of the theorem is implied by the Theorem 4. ■

Corollary 3. Let (X,d) be a complete metric space, $S: X \to X$ and $f \in \Theta$. If there exist $\alpha,\beta \geq 0$ such that $\alpha + 2\beta \in (0,1)$ and

$$f(d^2(Sx,Sy)) \leq (\alpha+\beta) f(d^2(x,Sx)) + \beta f(d^2(y,Sy)),$$

for all $x, y \in X$, then S has a unique fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to the fixed point.

Proof. The mapping $Tx = x$, for each $x \in X$ is continuous, injection and sequentially convergent. Therefore, the corollary is directly implied by Theorem 5 for $Tx = x$. ■

Corollary 4. Let (X,d) be a complete metric space, $S: X \to X$ and the mapping $T: X \to X$ is continuous, injection and subsequentially convergent. If there exist $\alpha,\beta \geq 0$ such that $\alpha + 2\beta \in (0,1)$ and

$$d^2(TSx,TSy) \leq (\alpha+\beta)d^2(Tx,TSx) + \beta d^2(Ty,TSy)$$

for all $x, y \in X$, then S has a unique fixed point.

Proof. The function $f(t) = t, t \geq 0$ is monotony increasing and $f^{-1}(0) = \{0\}$. Therefore, the corollary is a direct implication of Theorem 4 for $f(t) = t$. ■

Comment 2. 1) For $\alpha = 0$ and $\beta = \lambda$, in the Theorem 5 we get that in a complete metric space (X,d) if $S: X \to X$, $f \in \Theta$ and the mapping $T: X \to X$ is continuous, injection and subsequentially convergent and if it exists $\lambda \in (0,\frac{1}{2})$ is such that

$$f(d^2(TSx,TSy)) \leq \lambda(f(d^2(Tx,TSx)) + f(d^2(Ty,TSy)))$$

for all $x, y \in X$, then S has a unique fixed point.

2) If we take consideration that the mapping $Tx = x$, for all $x \in X$ is continuous, injection and subsequentially convergent, the Corollary 4 implies that if for the mapping $S: X \to X$ there exist $\alpha,\beta \geq 0$ such that $\alpha + 2\beta \in (0,1)$ and

$$d^2(Sx,Sy) \leq (\alpha+\beta)d^2(x,Sx) + \beta d^2(y,Sy), \tag{8}$$

for all $x, y \in X$, then S has a unique fixed point.

3) For $\alpha = 0$ and $\beta = \lambda$ in (8) we get that the Theorem5 implies the Theorem3.

Theorem 6. Let (X,d) be a complete metric space $S: X \to X$, the mapping $T: X \to X$ be continuous, injection and subsequentially convergent and $f \in \Theta$ is such that $f(a+b) \leq f(a) + f(b)$, for all $a,b \geq 0$. If there exist $\alpha > 0, \beta \geq 0$ such that $\alpha + 2\beta \in (0,1)$ and

$$f(d(TSx,TSy)) \leq (\alpha+\beta) f(d(Tx,TSy)) + \beta f(d(Ty,TSx)) \tag{9}$$

for all $x, y \in X$, then S has a unique fixed point.

Proof. Let x_0 be any point on X and the sequence $\{x_n\}$ be defined as the following $x_{n+1} = Sx_n$, $n = 0,1,2,3,...$. The inequality (9) and the property of f imply the followings

$$f(d(Tx_{n+1},Tx_n)) \leq \beta f(d(Tx_{n-1},Tx_n)) + \beta f(d(Tx_n,Tx_{n+1}))$$

and

$$f(d(Tx_{n+1},Tx_n)) \le (\alpha+\beta) f(d(Tx_{n-1},Tx_n)) + (\alpha+\beta) f(d(Tx_n,Tx_{n+1})) .$$

By summarizing the last two inequalities we obtain the following

$$f(d(Tx_{n+1},Tx_n)) \le \lambda f(d(Tx_n,Tx_{n-1})) , \tag{10}$$

for each $n = 1,2,3,...$, for $\lambda = \frac{\alpha+2\beta}{2-(\alpha+2\beta)} < 1$. Further, by applying the inequality (10), analogously as the proof in theorem 4, we get that the sequence $\{Tx_n\}$ is convergent. Therefore, the sequence $\{x_n\}$ consists of convergent subsequence, i.e. it exists $u \in X$ and a subsequence $\{x_{n(k)}\}$ of the sequence $\{x_n\}$ such that $\lim\limits_{k\to\infty} x_{n(k)} = u$. The continuity of T implies that $\lim\limits_{k\to\infty} Tx_{n(k)} = Tu$, that is $\lim\limits_{n\to\infty} Tx_n = Tu$. Further, the inequality (9), analogously as the proof in theorem 4, implies

$$f(d(TSu,Tx_{n+1})) \le \frac{\alpha+2\beta}{2}[f(d(Tu,Tx_{n+1})) + f(d(Tx_n,TSu))]$$

For $n \to \infty$ in the above inequality, the continuity of f and T and the properties of the metric, imply

$$f(d(TSu,Tu)) \le \frac{\alpha+2\beta}{2}[f(d(TSu,Tu)) + f(0)] .$$

Therefore, analogously as the proof in theorem 4, we conclude that $Su = u$, that is the mapping S has a fixed point.

Let $u,v \in X$ be fixed point on S, i.e. $Su = u$ and $Sv = v$. Then, (9) implies

$$\begin{aligned} f(d(Tu,Tv)) &= f(d(TSu,TSv)) \\ &\le (\alpha+\beta) f(d(Tu,TSv)) + \beta f(d(Tv,TSu)) \\ &= (\alpha+2\beta) f(d(Tu,Tv)). \end{aligned}$$

Therefore, $u = v$, i.e. S has a unique fixed point. Finally, if T e sequentially convergent, then by substituting the sequence $\{n(k)\}$ with the sequence $\{n\}$ and arbitrarily of $x_0 \in X$ and the above stated, imply that for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to the unique fixed point on S. ■

Corollary 5. Let (X,d) be a complete metric space, $S: X \to X$ and $f \in \Theta$ be such that $f(a+b) \le f(a) + f(b)$, for all $a,b \ge 0$. If it exist $\alpha,\beta \ge 0$ such that $\alpha + 2\beta \in (0,1)$ and

$$f(d(Sx,Sy)) \le (\alpha+\beta) f(d(x,Sy)) + \beta f(d(y,Sx)) ,$$

for all $x,y \in X$, then S has a unique fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to that point.

Proof. For $Tx = x$, in theorem 6 we get the corollary. ■

Corollary 6. Let (X,d) be a complete metric space, $S:X\to X$ and the mapping $T:L\to L$ be continuous, injection and subsequentially convergent. If it exists $\alpha,\beta\geq 0$ such that $\alpha+2\beta\in(0,1)$ and

$$d(TSx,TSy)\leq(\alpha+\beta)\,|\,d(Tx,TSy)+\beta d(Ty,TSx)$$

for all $x,y\in X$, then S has a unique fixed point.

Proof. For $f(t)=t$, in theorem 6 we get the corollary. ■

Comment 3. 1) For $\alpha=0$ and $\beta=\lambda$, the Theorem6 implies the Theorem 2.2 [2].

When expressing the Theorem 2.2 [2] is missing the condition of sub-additivity of the function $f\in\Theta$ which is applied at the beginning of the proof of the Theorem.

2) Having in mind that $Tx=x$, for all $x\in X$ is continuous, injection and subsequentially convergent, the Corollary 6 implies that if for the mapping $S:X\to X$ exist $\alpha,\beta\geq 0$ such that $\alpha+2\beta\in(0,1)$ and

$$d(Sx,Sy)\leq(\alpha+\beta)d(x,Sx)+\beta d(y,Sy), \tag{11}$$

for all $x,y\in X$, then S has a unique fixed point.

3) For $\alpha=0$ and $\beta=\lambda$ in (11), the Theorem6 implies the Theorem2.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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