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**SEVERAL RELATIONS BETWEEN THE ROOTS OF POLYNOMIALS,  
THE ROOTS OF THEIR DERIVATIVES AND  
THE FOCI OF IN-ELLIPSES**

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**Abstract.** It is a well known fact, that if the roots of a polynomial of third degree are in the vertices of a triangle, then the roots of its derivative are in the foci of the in-ellipse which is tangent to the sides of the triangle at their midpoints. This assertion motivates looking for answers to the following two basic questions: 1) Is there another type of polynomials that have roots in the vertices of a triangle, while their derivatives have roots in the foci of other in-ellipses of the triangle? 2) Is it possible to discover similar relations between the roots of polynomials in the vertices of polygons, the roots of their derivatives and the foci of in-ellipses of these polygons? We propose a short survey of some results which give answers to the above questions. We consider relations between polynomials with roots in the vertices of quadrilaterals or pentagons and the corresponding derivatives with roots in the foci of in-ellipses. It is noted a special kind of polynomials (medi-tangential) for which a relation exists between the roots of such a polynomial, the corresponding derivative and the foci of all in-ellipses. Thus, a generalization of the theorem for the triangles is obtained.

A remarkable relation between the roots of some polynomials of third degree, the roots of the corresponding derivatives and the foci of special ellipses is presented by the following theorem.

**Theorem 1.** *If the roots of a polynomial  $P(z)$  of third degree are in the vertices of a triangle  $\Delta$ , then the roots of its derivative  $P'(z)$  are in the foci of the in-ellipse which is tangent to the sides of  $\Delta$  at their mid-points.*

A proof of theorem 1 is presented in [1]. If  $P(z) = P(x)$  is a polynomial of third degree, of a real variable and with real coefficients, the result of theorem 1 has a geometric interpretation according to which the roots of  $P(x)$  are in the vertices of an isosceles triangle (Fig. 1).

Besides the ellipse that is mentioned in theorem 1, infinitely many ellipses could be inscribed in each triangle. For this reason a question arises about the possibility that some of the ellipses also realize geometric relations between polynomials and their corresponding derivatives. It turns out indeed, that some of these ellipses realize geometric relations between some kinds of polynomials with multiple roots in the vertices of a given triangle and the roots of their derivatives.

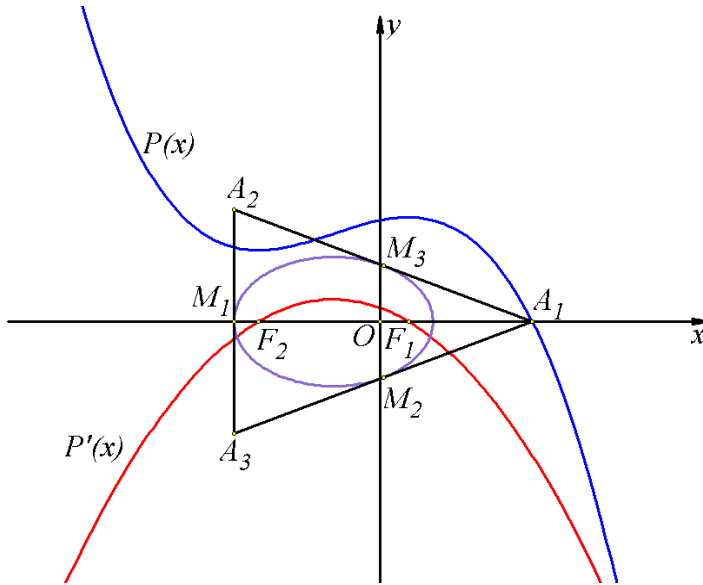


Fig. 1.

**Theorem 2.** *If a polynomial  $P(z)$  of degree  $n = k_1 + k_2 + k_3$  has a  $k_j$ -multiple root in the vertex  $A_j$  ( $j = 1, 2, 3$ ) of  $\Delta A_1 A_2 A_3$ , then the derivative  $P'(z)$  of  $P(z)$  has roots in the foci of the in-ellipse  $k$ , which is tangent to the lines  $A_2 A_3$ ,  $A_3 A_1$  and  $A_1 A_2$  at the points  $P_1$ ,  $P_2$  and  $P_3$ , respectively, verifying the equations*

$$\overline{A_2 P_1} : \overline{A_3 P_1} = -k_2 : k_3, \quad \overline{A_3 P_2} : \overline{A_1 P_2} = -k_3 : k_1, \quad \overline{A_1 P_3} : \overline{A_2 P_3} = -k_1 : k_2.$$

A proof of theorem 2 is presented in [2]. The polynomial  $P'(z)$  has a  $(k_j - 1)$ -multiple root in the vertex  $A_j$  ( $j = 1, 2, 3$ ) [3], while the other two roots are described by theorem 2. In such a way a full geometric picture of the roots of  $P'(z)$  is obtained.

The next consequence follows from theorem 2 directly:

**Corollary 1.** *If a polynomial  $P(z)$  of degree  $n = 3.m$  has a  $m$ -multiple root in the vertex  $A_j$  ( $j = 1, 2, 3$ ) of  $\Delta A_1 A_2 A_3$ , then the derivative  $P'(z)$  of  $P(z)$  has roots in the foci of the in-ellipse  $k$ , which is tangent to the segments  $A_2 A_3$ ,  $A_3 A_1$  and  $A_1 A_2$  at their mid-points.*

It is obvious, that theorem 1 follows from corollary 1 when  $m = 1$ .

If  $P(z) = P(x)$  is a polynomial of a real variable and with real coefficients, then some geometric interpretations of theorem 2 and its consequence could be presented. In fig. 2 the polynomial  $P(x)$  has a double root in  $A_1$  and two

simple complex conjugated roots in  $A_2$  and  $A_3$ . The derivative  $P'(x)$  has three simple real roots in the vertex  $A_1$  and the foci  $F_1$  and  $F_2$  of the ellipse  $k$ . In fig. 3 the polynomial  $P(x)$  has a simple real root in  $A_1$  and two double complex conjugated roots in  $A_2$  and  $A_3$ . The derivative  $P'(x)$  has two simple real roots in the foci  $F_1$  and  $F_2$  of the ellipse  $k$ . The polynomials, which correspond to the corollary are presented in fig. 4 and 5. These polynomials have double and 3-multiple roots in the vertices of  $\Delta A_1 A_2 A_3$ , correspondingly. They are obtained when  $m = 2$  and  $m = 3$ .

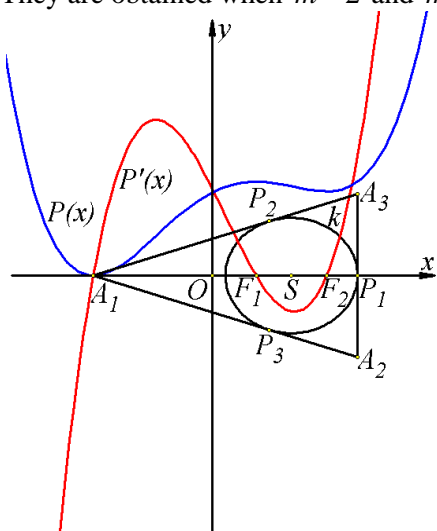
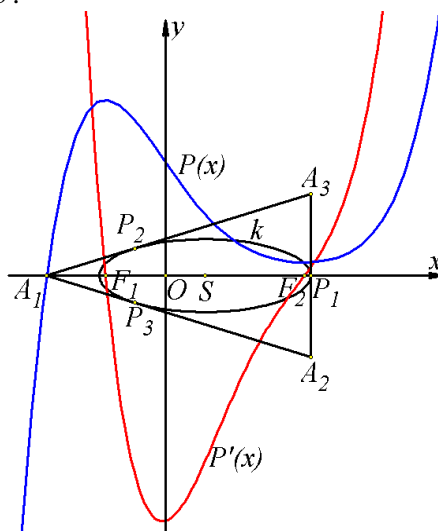


Fig. 2



6Fig. 3

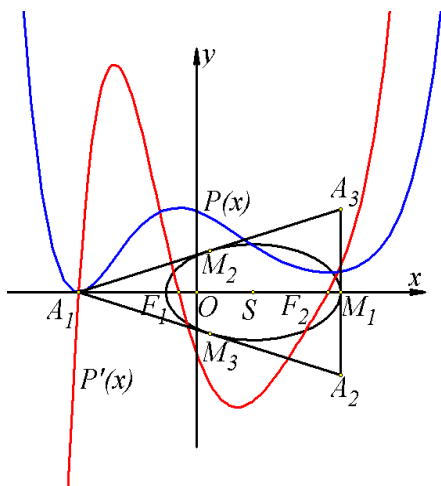


Fig. 4

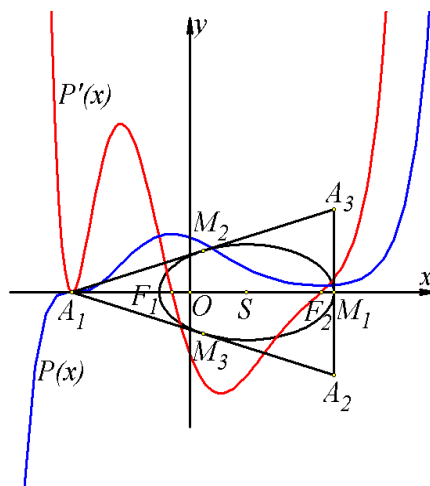


Fig. 5

It seems, that among the other geometric figures the parallelogram is mostly close to the triangle. There is an ellipse in the parallelogram too, which is tangent at the mid-points of its sides. Thus, the following proposal appears in a natural way;

**Theorem 3.** *If the roots of a polynomial  $P(z)$  of fourth degree are in the vertices of a parallelogram  $P$ , then the roots of the derivative  $P'(z)$  of  $P(z)$  are in the foci and the center of the ellipse, which is tangent to the sides of  $P$  at their mid-points.*

A proof of theorem 3 is presented in [4]. If  $P(z) = P(x)$  is a polynomial of fourth degree of a real variable and with real coefficients, then a geometric interpretation of theorem 3 exists. It could be applied if the polynomial under consideration is one of the following two kinds:

- 1)  $P(x)$  has two real and two complex conjugated roots. This means that the parallelogram is a rhombus;
- 2)  $P(x)$  has two pairs complex conjugated roots. This means that the parallelogram is a rectangle.

In case 1) the graphical representation of  $P(x)$ , as an image of a function of a real variable, passes through two opposite vertices of the rhombus only, which corresponds to the two real roots of  $P(x)$ . The graphical representation of  $P'(x)$  is a cubic parabola whose center of symmetry is in the intersection point  $S$  of the diagonals of  $A_1A_2A_3A_4$ , which is always on the abscise axis.

The ellipse  $\kappa$ , which is tangent to the sides of  $A_1A_2A_3A_4$  at their mid-points, has foci on the abscise axis or on the diagonal of  $A_1A_2A_3A_4$ , parallel to the ordinate axis. In the first case the cubic parabola  $P'(x)$  intersects the abscise axis in the center  $S$  and the foci  $F_1$  and  $F_2$  of  $\kappa$ . In this case the roots of  $P'(x)$  are real and coincide with the real points  $S$ ,  $F_1$  and  $F_2$  (Fig. 6). In the second case the cubic parabola  $P'(x)$  intersects the abscise axis in the point  $S$  only and does not pass through the points  $F_1$  and  $F_2$ . In this case the points  $F_1$  and  $F_2$  correspond to non-real roots of  $P'(x)$ . However, always the point  $S$  corresponds to a real root and for this reason the graphical representation of  $P'(x)$  passes through it always (Fig. 7).

In case 2) the graphical representation of  $P(x)$ , as an image of a function of real variable, does not pass through any vertex of the rectangle  $A_1A_2A_3A_4$  in the general case. The graphical representation of  $P'(x)$  is a cubic parabola with center of symmetry in the intercept point  $S$  of the diagonals of  $A_1A_2A_3A_4$ , which is on the abscise axis always.

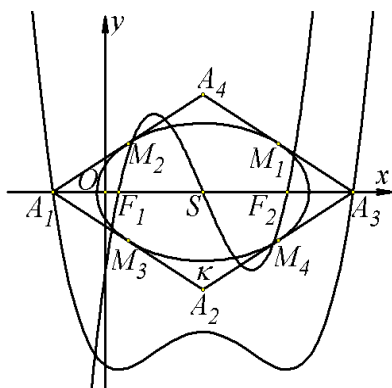


Fig.

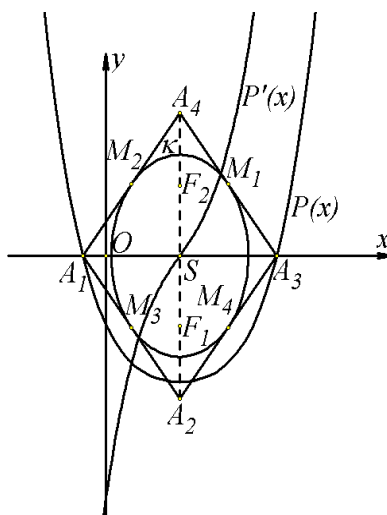


Fig. 7

The ellipse  $\kappa$ , which is tangent to the sides of  $A_1A_2A_3A_4$  at their mid-points, has foci on the abscise axis or on a line, parallel to the ordinate axis.

In the first case the cubic parabola  $P'(x)$  intersects the abscise axis in the center  $S$  and the foci  $F_1$  and  $F_2$  of  $\kappa$ . In this case the roots of  $P'(x)$  are real and coincide with the real points  $S$ ,  $F_1$  and  $F_2$  of  $\kappa$  (Fig. 8).

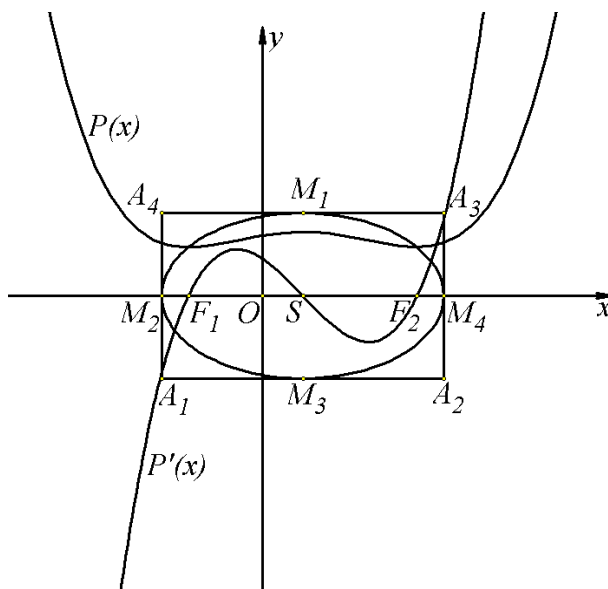
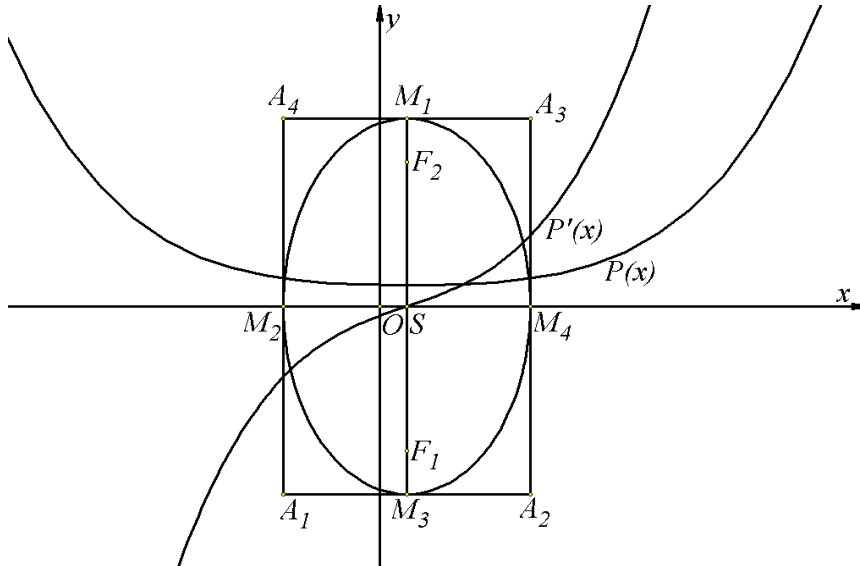


Fig. 8

In the second case the cubic parabola  $P'(x)$  intersects the abscise axis in the point  $S$  only and does not pass through the points  $F_1$  and  $F_2$ . In this case the points  $F_1$  and  $F_2$  correspond to non-real roots of  $P'(x)$ . However, always the point  $S$  corresponds to a real root and for this reason the graphical representation of  $P'(x)$  passes through it always (Fig. 9).



**Fig. 9**

Besides the ellipse that is mentioned in theorem 3, infinitely many ellipses could be inscribed in each parallelogram. A question arises about the possibility that some of the ellipses also realize geometric relations between polynomials and their corresponding derivatives. It turns out indeed, that some of these ellipses realize geometric relations between some kinds of polynomials with multiple roots in the vertices of a given parallelogram and the roots of their derivatives.

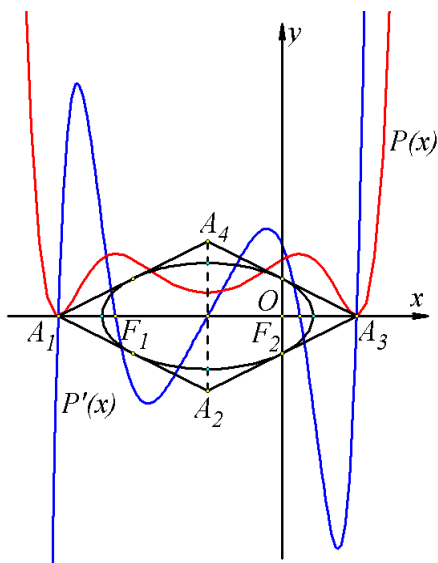
**Theorem 4.** *If a polynomial  $P(z)$  of degree  $n = 2 \cdot (k_1 + k_2)$  has  $k_1$ -multiple roots in the vertices  $A_1$  and  $A_3$ , also  $k_2$ -multiple roots in the vertices  $A_2$  and  $A_4$  of the parallelogram  $A_1A_2A_3A_4$ , then the derivative  $P'(z)$  of  $P(z)$  has roots in the foci and the center of the ellipse  $k$ , which is tangent to the lines  $A_3A_4$ ,  $A_4A_1$ ,  $A_1A_2$  and  $A_2A_3$  at the points  $P_1$ ,  $P_2$ ,  $P_3$  and  $P_4$ , respectively, verifying the equalities*

$$\begin{aligned} \overline{A_3P_1} : \overline{A_4P_1} &= -k_1 : k_2, & \overline{A_4P_2} : \overline{A_1P_2} &= -k_2 : k_1, \\ \overline{A_1P_3} : \overline{A_2P_3} &= -k_1 : k_2, & \overline{A_2P_4} : \overline{A_3P_4} &= -k_2 : k_1. \end{aligned}$$

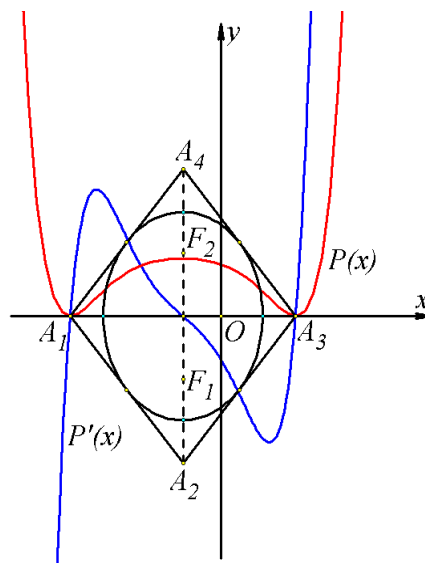
A proof of theorem 4 is presented in [5].

**Corollary 2.** *If a polynomial  $P(z)$  of degree  $n=4.m$  has a  $m$ -multiple root in the vertex  $A_j$  ( $j=1,2,3,4$ ) of the parallelogram  $A_1A_2A_3A_4$ , then the derivative  $P'(z)$  of  $P(z)$  has roots in the foci and the center of the ellipse  $k$ , which is tangent to the segments  $A_3A_4$ ,  $A_4A_1$ ,  $A_1A_2$  and  $A_2A_3$  at their midpoints.*

It is clear, that when  $m=1$  corollary 2 implies theorem 1. If  $P(z)=P(x)$  is a polynomial with real coefficients of a real variable, then some geometric interpretations of the corollary could be presented. Fig. 10 and Fig. 11 present polynomials  $P(x)$  with double roots in the vertices of rhombus  $A_1A_2A_3A_4$ . In the case of Fig. 10 the derivative  $P'(x)$  has real roots in the foci  $F_1$ ,  $F_2$  and the center  $O$  of the ellipse  $k$ , which is tangent at the mid-points of the sides  $A_3A_4$ ,  $A_4A_1$ ,  $A_1A_2$  and  $A_2A_3$ . For this reason the graphical representation of  $P'(x)$  passes through the points  $F_1$ ,  $F_2$  and the center of  $k$ . In the case of Fig. 11 the derivative  $P'(x)$  has no real root in the foci  $F_1$  and  $F_2$  of the ellipse  $k$ . For this reason the graphical representation of  $P'(x)$  passes through the center of  $k$  only.



**Fig. 10**



**Fig. 11**

Fig. 12 and Fig. 13 present polynomials  $P(x)$  with 3-multiple roots in the vertices of the rectangle  $A_1A_2A_3A_4$ . In the case of Fig. 12, the derivative  $P'(x)$  has real roots in the foci  $F_1$ ,  $F_2$  and the center of the ellipse  $k$ , which is tangent at the mid-points of the sides  $A_3A_4$ ,  $A_4A_1$ ,  $A_1A_2$  and  $A_2A_3$ . For this

reason the graphical representation of  $P'(x)$  passes through the points  $F_1, F_2$  and the center of  $k$ . In the case of Fig. 13, the derivative  $P'(x)$  has no real root in the foci  $F_1$  and  $F_2$  of the ellipse  $k$ . For this reason the graphical representation of  $P'(x)$  passes through the center of  $k$  only.

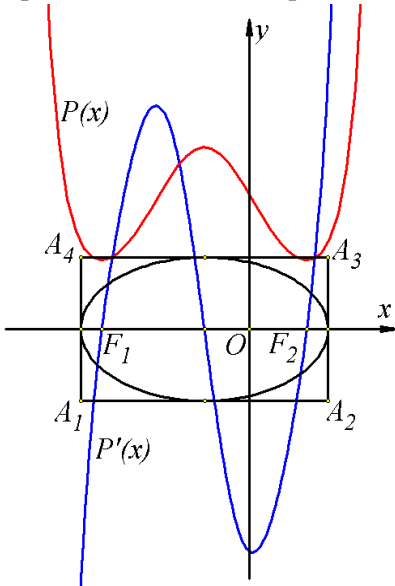


Fig. 12

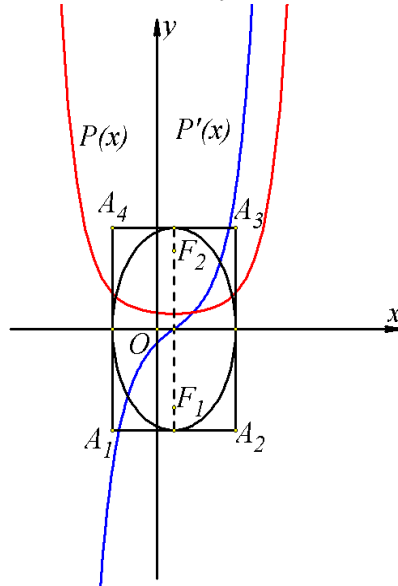


Fig. 13

The geometric relation presented in theorem 4, contains the foci of a suitable in-ellipse of a parallelogram. On the other hand infinitely many ellipses could be inscribed in each convex quadrilateral. Thus, a question arises for the existence of a relation between some of those ellipses and the derivatives of the polynomials with roots in the vertices of a given quadrilateral only. Such a relation is presented in the following assertion.

**Theorem 5.** Let  $k_j$  ( $j=1,2,3,4$ ) be positive integers and  $k$  be an in-ellipse of the convex quadrilateral  $A_1A_2A_3A_4$ , such that the tangent points  $P_1, P_2, P_3$  and  $P_4$  of  $k$  with the segments  $A_1A_2, A_2A_3, A_3A_4$  and  $A_4A_1$ , respectively, verify the equalities:

$$\overline{A_1P_1} : \overline{A_2P_1} = -k_1 : k_2, \quad \overline{A_2P_2} : \overline{A_3P_2} = -k_2 : k_3,$$

$$\overline{A_3P_3} : \overline{A_4P_3} = -k_3 : k_4, \quad \overline{A_4P_4} : \overline{A_1P_4} = -k_4 : k_1.$$

If a polynomial  $P(z)$  of degree  $n=k_1+k_2+k_3+k_4$  and complex variable, with complex coefficients has a  $k_j$  multiple root in the vertices  $A_j$  ( $j=1,2,3,4$ ) of  $A_1A_2A_3A_4$ , then the derivative of  $P(z)$  has roots in the point  $P_0 = A_1A_3 \cap A_2A_4$  and in the foci of the ellipse  $k$ .

A proof of theorem 5 is presented in [6]. The polynomial  $P'(z)$  has a  $k_j - 1$  multiple root in the vertex  $A_j$  ( $j=1,2,3,4$ ) [3], while the other three roots are described by theorem 5. In such a way we obtain a full geometric picture of the roots of  $P'(z)$ . Of course, theorem 5 is a generalization of theorem 4.

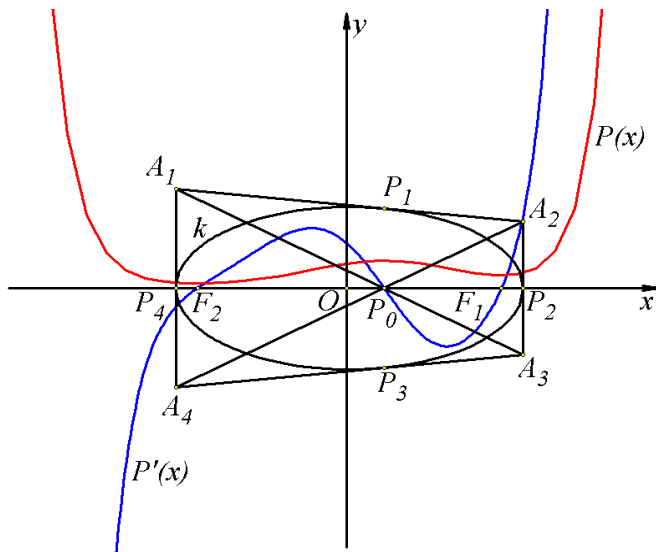


Fig. 14

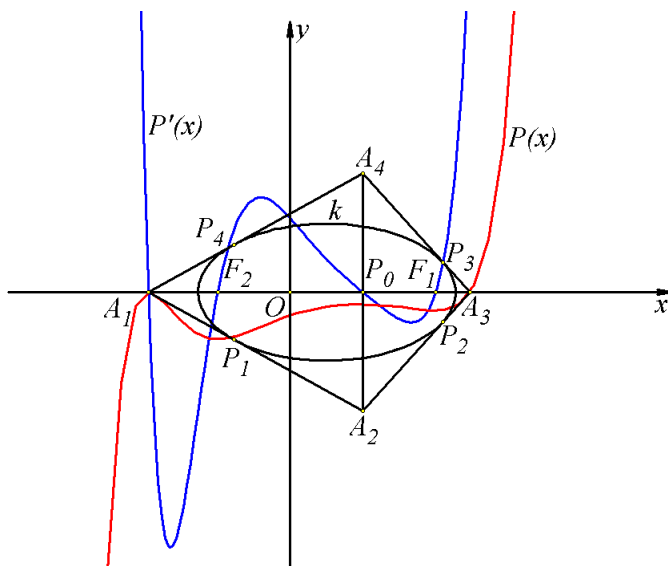


Fig. 15

If  $P(z) = P(x)$  is a polynomial with real coefficients, of real variable, we could present some geometric interpretations of the proved theorem. Fig. 14 presents a polynomial  $P(x)$  with roots in the vertices of a trapeze, which is symmetric with respect to the abscise axis. It has 3-multiple complex conjugated roots in the points  $A_1, A_4$  and double complex conjugated roots in the points  $A_2$  and  $A_3$ . Fig. 15 presents a polynomial  $P(x)$  with roots in the vertices of a deltoid, which is symmetric with respect to the abscise axis. It has a double real root in the point  $A_1$ , a simple real root in the point  $A_3$  and a 3-multiple complex conjugated roots in the points  $A_2$  and  $A_4$ .

The geometric relation described in theorem 5, contains the foci of a suitable in-ellipse of the quadrilateral under consideration (if such an ellipse exists). On the other hand, each convex pentagon  $A_1A_2A_3A_4A_5$  contains exactly two ellipses – the one is inscribed in  $A_1A_2A_3A_4A_5$ , while the other one is inscribed in the star-like pentagon  $A_1A_3A_5A_2A_4$ . In such a way a question arises for the existence of a relation between some kinds of convex pentagons and their in-ellipses in comparison with the derivatives of the polynomials with roots in the vertices of the pentagons under consideration only. It turns out that such a relation exists, which is similar to the one for quadrilaterals. It is expressed by the following assertion.

**Theorem 6.** *Let  $k_j$  ( $j=1,2,3,4,5$ ) be positive integers and  $k'$  be an in-ellipse of the convex pentagon  $A_1A_2A_3A_4A_5$ , such that the tangent points  $P_1, P_2, P_3, P_4$  and  $P_5$  of  $k'$  with the segments  $A_1A_2, A_2A_3, A_3A_4, A_4A_5$  and  $A_5A_1$ , respectively, verify the equalities:*

$$\overline{A_1P_1} : \overline{A_2P_1} = -k_1 : k_2, \overline{A_2P_2} : \overline{A_3P_2} = -k_2 : k_3, \overline{A_3P_3} : \overline{A_4P_3} = -k_3 : k_4,$$

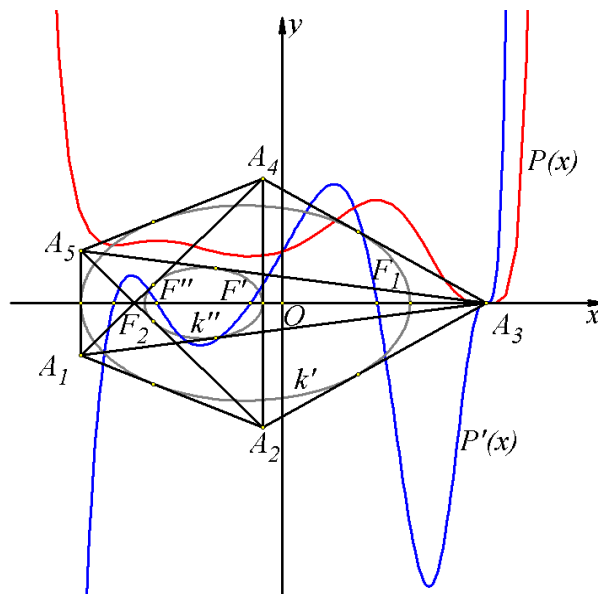
$$\overline{A_4P_4} : \overline{A_5P_4} = -k_4 : k_5, \overline{A_5P_5} : \overline{A_1P_5} = -k_5 : k_1.$$

*If the polynomial  $P(z)$  of degree  $n = k_1 + k_2 + k_3 + k_4 + k_5$  has a  $k_j$  multiple root in the vertices  $A_j$  ( $j=1,2,3,4,5$ ) of  $A_1A_2A_3A_4A_5$ , then the derivative of  $P(z)$  has roots in the foci of the ellipse  $k'$  and the foci of the ellipse  $k''$ , which is inscribed in the star-like pentagon  $A_1A_3A_5A_2A_4$ .*

A proof of theorem 6 is presented in [7]. The polynomial  $P'(z)$  has a  $k_j - 1$  multiple root in the vertex  $A_j$  ( $j=1,2,3,4,5$ ) [3], while the other four roots are described by theorem 6. In such a way we obtain a full geometric picture of the roots of  $P'(z)$ .

If  $P(z) = P(x)$  is a polynomial with real coefficients of a real variable we could present some geometric interpretations of theorem 6. Fig. 16 presents a polynomial  $P(x)$  with roots in the vertices of a pentagon, which is symmetric

with respect to the abscise axis. It has double complex conjugated roots in the points  $A_1$  and  $A_5$ , 3-multiple complex conjugated roots in the points  $A_2$  and  $A_4$  and a real root in the point  $A_3$ .



**Fig. 16**

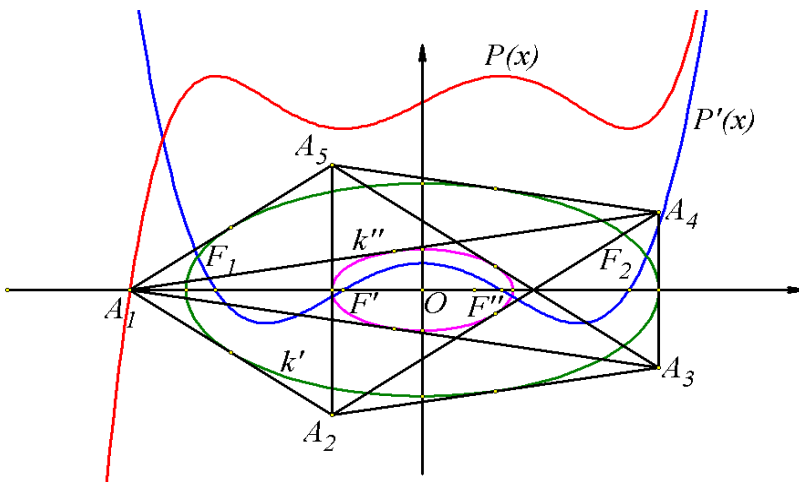
Theorems 1 and 2 formulated before concern polynomials of third and fourth degree, located in the vertices of a triangle and a parallelogram, respectively. They could be generalized considering a special class of polygons, which includes the triangle and the parallelogram. This class consists of polygons which are simultaneously inscribed and exscribed for concentric homothetic ellipses. It is determined in the following way: Let  $A_1A_2 \dots A_n$  ( $n \geq 3$ ) be an arbitrary convex polygon for which  $B_1, B_2, \dots, B_{n-1}, B_n$  are the mid-points of the sides  $A_1A_2, A_2A_3, \dots, A_{n-1}A_n, A_nA_1$ , respectively. If  $A_1A_2 \dots A_n$  has an in-ellipse  $k$ , such that the tangent points of  $k$  with the sides of  $A_1A_2 \dots A_n$  are  $B_j$  ( $j=1,2,\dots,n$ ), we will call  $A_1A_2 \dots A_n$  to be *meditangential polygon* of  $k$ , while we will call  $k$  to be *meditangential ellipse* for  $A_1A_2 \dots A_n$ . The following theorem is true for the *meditangential polygons*.

**Theorem 7.** *If a polynomial  $P(z)$  of degree  $n$  has roots in the vertices of a meditangential  $n$ -polygon  $A_1A_2 \dots A_n$ , then the roots of the derivative  $P'(z)$  of  $P(z)$  are in the foci of all meditangential ellipses, generated by the vertices of  $A_1A_2 \dots A_n$ , and in the center of  $A_1A_2 \dots A_n$ , when  $n$  is even.*

A proof of theorem 7 is presented in [8]. This theorem implies directly the following:

**Corollary 3.** *If the roots of the polynomial  $P(z)$  are in the vertices of a mediant polygon, then the roots of its derivative  $P'(z)$  are located on a straight line.*

**Corollary 4.** *If a polynomial  $P(z)$  of degree  $n.p$  has a  $p$  multiple root in the vertex  $A_j$  ( $j=1,2,\dots,n$ ) of a mediant polygon  $A_1A_2\dots A_n$ , then the derivative  $P'(z)$  of  $P(z)$  has roots in the foci of all mediant ellipses, generated by the vertices of  $A_1A_2\dots A_n$  and in the center of  $A_1A_2\dots A_n$ , when  $n$  is even.*



**Fig. 17**

If  $P(z)=P(x)$  is a polynomial with real coefficients of real variable we could present some geometric interpretations of theorem 7. Fig. 17 presents a polynomial  $P(x)$  with roots in the vertices of a *mediant pentagon*, which is symmetric with respect to the abscise axis. Fig. 18 presents a polynomial  $P(x)$  with roots in the vertices of a *mediant pentagon*, symmetric with respect to the abscise axis, which has a six-multiple real root in the point  $A_1$ , six multiple complex conjugated roots in the pairs of points  $(A_2, A_5)$  and  $(A_3, A_4)$ . This case demonstrates corollary 4.

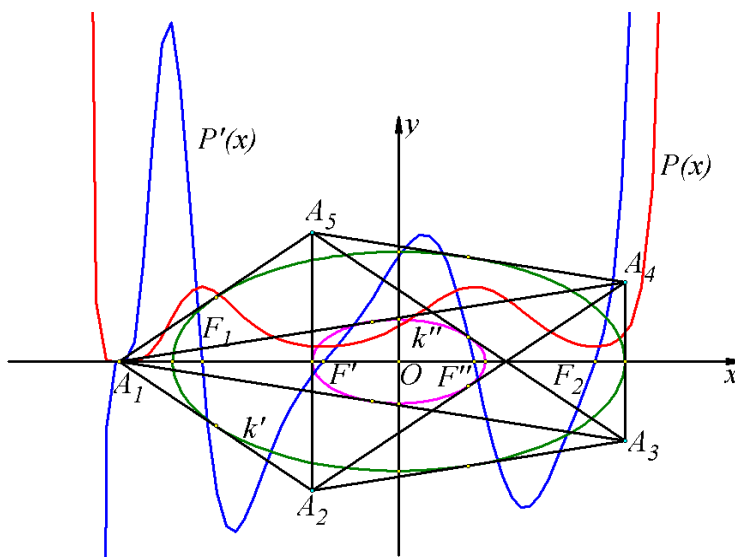


Fig. 18

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$$1) \int \frac{\sqrt{x} dx}{(a \pm bx)^{m+1}}$$

$$\int \frac{x\sqrt{x} dx}{a - bx} = \frac{6a\sqrt{x} - 2bx}{3b^2}$$

$$\frac{a - x + x\sqrt{x}}{(a - bx)^{m+1}} + \frac{3}{2(m-1)}$$

$$= \frac{2a\sqrt{x} + \frac{a\sqrt{a}}{b^2\sqrt{b}} \ln \left| \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}} \right|}{2(m-1)}$$