Union of Mathematicians of Macedonia - ARMAGANKA

# IX SEMINAR OF DIFFERENTIAL EQUATIONS AND ANALYSIS

and

1st CONGRESS OF DIFFERENTIAL EQUATIONS, MATHEMATICAL ANALYSIS AND APPLICATIONS

# CODEMA 2020

## Proceedings of the CODEMA 2020
## Зборник на трудови од CODEMA 2020

Skopje, 2021

# MAXIMUM RELIABILITY *K*-CENTER LOCATION PROBLEM

ISBN 978-608-4904-09-0 UDC: 519.872/.874:519.178

Nataša Kontrec, Biljana Panić, Marina Tošić, Mirko Vujošević

**Abstract.** This paper presents an approach for solving the maximum reliability *k*-center location problem. We are modifying the well-known *p*-center problem in order to determine the location of the observed objects and maximize the reliability of supply system coverage. The problem is defined as a stochastic problem of multi-center location on a graph. As a solution, two new algorithms have been proposed. The first is modified Dijkstra's algorithm for determination of the most reliable paths between nodes in the graph. The output of this algorithm is used as an input for the second algorithm designed to find the reliability of node coverage from a predetermined number of nodes.

## 1. INTRODUCTION

There are numerous mathematical models formulated in order to solve complex location problems. Some of these models are described in [1]-[3]. In this paper, we are observing the location model with stochastic input data. We present an algorithm for calculating the performances of a system as a basis for finding the optimal location. Most of the models described in the existing literature are deterministic but the practice showed that it is necessary to include stochasticity in the facility's location planning. Therefore, in this paper, a stochastic model of *k*-centre location on the graph has been formulated to determine the location of a given number of facilities to maximize the reliability of the system. The problem refers to a network structure that is determined by a graph whose nodes contain the locations of demand and potential facilities, while the weight of the branches represents reliability, i.e. the probability that an appropriate branch is available (operational). In the end, a new algorithm has been formulated to determine the reliability of covering a node from *k* nodes (*k* -covering reliability).

The location model with stochastic inputs has been studied in various situations. A special case of a stochastic set-covering location problem was studied in [4], while Alegre et al. [5] solved a stochastic facility location problem for determining the best locations of health resources for patients who have suffered a diabetic coma. The location problem with stochastic demand was considered in [6]-[7]. Some of the researchers used the scenario planning in order to include stochasticity in location planning [8]-[9]. Classical facility location models assume that once constructed, the chosen facilities will always operate as planned. In reality, facilities fail from time to time.

---

2010 *Mathematics Subject Classification.* 90B15; 90B25
*Key words and phrases*. location, graph, reliability, supply chain

## 2. PROBLEM DESCRIPTION

In this paper we are discussing the location problems on networks. Actually, we are observing the weighted graph in which each branch is joined with the number from the interval (0, 1). That number presents the reliability of the branch i.e. the probability that the information will be transmitted or the flow of the vehicle through a traffic line will be achieved during a certain period without failure. So, we assume that the failure of the branches are jointly statistically independent events and based on that, the reliability of the path between two nodes can be defined as the product of reliability of branches of which this path consists. Considering that there can be several paths between two nodes in a graph, the probability of jointly reaching a pair of nodes is defined as the probability of the most reliable path between the observed nodes. This probability can alternatively be called the reliability of covering the terminal node from the given initial node. The probability of mutual reaching defined in such a manner differs from the commonly adopted definition of reliability between two nodes of a network structure, which takes into account all of the possible paths between them.

In order to select the optimal location for the facility, we set up the classic problem of the maximum coverage of the remaining nodes from the selected one. Furthermore, the nodes which location needs to be determined, we will denote as warehouses, while for the nodes that need to be reached form the warehouses, we will use the term - consumers. So, a max-min optimization task is formulated and the lowest value of calculated probabilities is used as a criterion.

In order to solve the problem of the facilities (warehouse) location, we assumed that the consumer is covered from at least one facility and coverage (reaching) of the consumer from a specific warehouse is achieved along the path of maximum reliability from that particular warehouse to the consumer.

Based on the given assumptions, it follows that the probability of covering a consumer should be calculated as the probability of a union of events that the consumer will be covered from the observed warehouses. Starting from this definition, analogous to the location problem of a single warehouse, it is again possible to calculate the probability of covering every consumer and to adopt the maximal as the criterion of optimization.

Mathematically, this problem can be described as:

$$(\max) \sum_{(i,j)\in E} r_{ij} x_{ij}$$

s.t.

$$\sum_{j\in \Gamma(c)} x_{cj} = 1, \qquad \sum_{j\in \Gamma^{-1}(t)} x_{jt} = 1,$$

$$\sum_{j\in \Gamma^{-1}(i)} x_{ji} = \sum_{j\in \Gamma(i)} x_{ij}, \quad i \in V \setminus \{c,t\},\ x_{ij} \in \{0,1\}, \quad (i,j) \in E$$

$r_{ij}$, $(i, j) \in V$ are reliabilities assigned to each branch of the observed weighted graph and $x \in \{0,1\}^E$. Constraints should ensure that the obtained solution is indeed the elementary path of maximum reliability between start node (consumer) *c* and end node *t*.

## 3. PROBLEM SOLVING

In order to determine the reliability of covering a node from *k* location nodes $R_i$ an algorithm given below is defined. The paths of maximum reliability $R_{iw}$ between consumer and each of the *k* warehouses need to be found. All those most reliable paths make a set $V_{iw}$ and form a subgraph (oriented graph with a direction from its root towards its leaves). The locations of the warehouses are randomly determined and for each consumer, the path of maximal reliability between it and each of the *k* warehouses has been found.

First, we randomly determine the *k* location (of nodes) in which the facilities (warehouses) will be located. For every consumer, it is necessary to find the elementary path of maximal reliability between it and each of the *k* warehouses. To achieve that we are using the modified Dijkstra's algorithm as follows:

ALGORITHM 1 - ALGORITHM FOR DETERMINING THE PATH OF MAXIMUM RELIABILITY

1. Initial labels are joined to nodes as follows:

– The first node *c* is denoted $p^+(c) = 1$ (it gets a permanent label);

– Remaining nodes get temporary labels: $p^-(j) = 0, \quad \forall j \in V \setminus \{c\}$;

– *i* gets a value: $i = c$

2. We determine a set $A_i$ of the nodes following node *i*, which do not have a permanent label:

$$A_i = \left\{ j \,\middle|\, j \in \Gamma(i) \wedge p(j) = p^-(j) \right\}.$$

3. For each $j \in A_i$, new temporary labels are determined as:

$$p^-(j) = \max\left\{ p^-(j), p^+(i)^* r_{ij} \right\}.$$

4. Only one node $j^*$ receives a permanent label (out of all temporarily labelled), and it is the one for which:

$$p(j^*) \to p^+(j^*); \;\; j^* : p^-(j) = \max_{j \in N}\left\{ p^-(j) \right\}$$

so $p^+(j^*) = p^-(j^*)$.

5. If $j^* = t$, i.e. the ending node is marked with a permanent label. If not, then $i = j^*$ and return to step 2. If it is, the reliability of the path $p^*(t)$ has been determined.
6. Moving backwards for node *t* towards node *c*: $t \to j_k \to j_{k-1} \to ... \to c$ the path of maximal reliability $p_{\max} = (c, j_1, j_2, ..., j_k, t)$ is determined:

$$j_k : p^+(t) / p^+(j_k) = r_{j_k t}$$

$$j_{k-1} : p^+(j_k) / p^+(j_{k+1}) = r_{j_{k-1} j_k}$$

$$...$$

$$c : p^+(j_1) / p^+(c) = r_{cj_1}$$

7. It is evident that:

$$p^+(t) = r_{cj_1} * r_{j_1 j_2} * ... * r_{j_{k-1} j_k} * r_{j_k t}.$$

By using this algorithm, all the paths $P_{iw}, w \in W, i \in I$ are determined, we should determine reliability $R_i$ of covering consumer *i* from all *k* warehouses. To solve this task for particular consumer *i*, we firstly form a tree structure graph of all of the paths $P_w, \quad w \in W$, where root is a consumer *i* and the leaves are warehouse nodes.

It can be noticed that one maximum reliability path from a warehouse $w_1$ to a particular customer *i* can include a node that represents a different warehouse $w_2$. In this case, it is reasonable to exclude from consideration the sub-path between warehouses.

For the disjoint paths $P_{iw}$ the *k*-covering reliability is calculated based on a well-known formula:

$$R_i = \sum_{w=1}^{k} R_{iw} - \sum_{w=1}^{k-1} \sum_{j=w+1}^{k} R_{iw} R_{ij} + \sum_{w=1}^{k-2} \sum_{j=w+1}^{k-1} \sum_{z=j+1}^{k} R_{iw} R_{ij} R_{iz} - ... + (-1)^{k-1} \prod_{w=1}^{k} R_{iw}$$

Furthermore, the novel algorithm has been developed for calculating reliability $R_i$, which is based on Bellman's principle of optimality and concept of notation.

Let $V_i$ represent the set of all nodes of the tree. Each node from the set $V_i$ is joined with the notation $r_j$ which can be temporary $r_j^-$ or permanent $r_j^+$. Permanent notation of the node $j \in V_i$ represents the (maximum) reliability of covering node *j* from the observed warehouses, i.e. leaves of the tree. (By definition, the reliability of covering of the leaves (warehouses) equals 1. $r_w = 1 \quad \forall w \in W$ ) The temporary notation of node *j* is the temporary reliability of covering node *j*. It is always less than or equal to the permanent notation. In addition to these, the following notations shall be used:

$A_j$ – set of the successors of node *j* $A_j = \Gamma(j)$

$B_l$ – set of the predecessors of node *l* $B_l = \Gamma^{-1}(l)$

*S* – set of nodes with a permanent notation

*Q*– set of nodes with a temporary notation

The sets *S* and *Q* are iteratively exchanged.

The algorithm for calculating the reliability of covering a node (*k*– covering reliability) consists of the following steps:

ALGORITHM 2 - ALGORITHM FOR DETERMINATION OF RELIABILITY OF COVERING A NODE FROM *K* NODE

1. Initialization

   $S = W$

   $Q = V_i \setminus S$

   $r_w = r_w^+ = 1 \quad \forall w \in W$

   $r_j = r_j^- = 0 \quad \forall j \in Q$

   $A_j = \left\{ l \,\middle|\, l \in \Gamma(j) \wedge l \in V_i \right\} \quad \forall j \in V_i$

2. Out of the set of unlabelled nodes *Q*, determine the set of nodes whose successors are all permanently labelled. First, determine the set of all nodes which precede the permanently labelled nodes:

   $$B = \left\{ l \,\middle|\, l \in \Gamma^{-1}(S) \wedge \forall l \in Q \right\},$$

   then out of those, take the nodes whose successors are all permanently labelled:

   $$A = \left\{ j \,\middle|\, j \in B_i \wedge \Gamma(j) \in S \right\}$$

3. Calculate the reliability of node *j*:

   $$r_j = \sum_{(l \in A_j)} r_{jl} r_l - \sum_{(l \in A_j)} \sum_{(k \in A_j)} r_{jl} r_l r_{jk} r_k +$$

   $$\sum_{(l \in A_j)} \sum_{(k \in A_j)} \sum_{(z \in A_j)} r_{jl} r_l r_{jk} r_k r_{lz} r_z - \ldots + (-1)^{M_i - 1} \prod_{(t \in A_j)}^{M_i} r_{jt} r_t \quad \forall j \in A$$

   Change the notations of these nodes into permanent ones: $r_j = r_j^+$

4. Update the set of permanently labelled nodes: $S = S \cup A$
5. Determine the set of unlabelled nodes: $Q = V_i \setminus S$
6. If $Q \neq \varnothing$, return to step 2. If it is equal, proceed to step

The end.

As we can see, this problem can be observed as a sub-problem of the problem of determining the location of *k* warehouse in order to maximize system reliability. The reliability of the system is equal to the minimal reliability of covering nodes that comprise it. The procedure for calculation of system reliability remains the same if we chose some other *k* location for the warehouses. If we were to examine all possible warehouse locations, then we would select the particular combination of *k* warehouses for which the reliability of the system would be maximal.

## 5. CONCLUSION

As a result, we showed how it is possible to determine the most reliable paths between any two nodes of a graph and determine their reliability by using the modification of Dijkstra's algorithm. Also, system reliability can be calculated by determining the reliability of node coverage. Variables of this problem are integers, so this is a combinatorial problem. The objective function is not given analytically, so it has to be calculated numerically. Constraints arise from topological characteristics and define locations on the graph. The only constraint is a given number of nodes i.e. facilities. For further research, we will consider to formulate a relaxed problem and solve it by using some of the available software or to develop a heuristic algorithm or to apply some of the metaheuristics.

## NOTATION

$G=(V,E,R)$ – weighted graph, where:

$V=(1,...,i,...,n)$ – set of nodes

$E\subseteq V\times V=\{(i,j)|i\in V, j\in V\}$ - set of arcs

*R* - function which joins to each arc (*i, j*) the weight (reliability) $r_{ij}$ from the interval $[0,1]$ $R:E\to[0,1]$

$r_{ij}$ - reliability of arc (*i* , *j*).

$W=\{j_1,..., j_w,..., j_k\}$ – set of warehouse indexes, $W\subset V$

*k* – total number of warehouses to be located

$C=V\setminus W=\{j_{k+1},..., j_i,..., j_n\}$ – set of consumer nodes

For the purpose of simplicity, warehouse indexes will be abbreviated as *w*, $w\in W$ while consumer indexes as *i*, $i\in C$

$P_{iw}$ – path of maximum reliability from warehouse *w* to consumer *i*. The path consists of an array of arcs $((w,v_{j1}),(v_{j1},v_{j2}),...,(v_{jl},i))$ and is alternatively represented as an array of nodes $P_{iw}=(w,v_{j1},v_{j2},...,v_{jl},i)$.

$R_{iw}$ – reliability of covering consumer from warehouse w (reliability of path $P_{iw}$) $R_{iw} = \prod_{(j,l)\in P_{iw}} r_{jl}$

$R_i$ – reliability of covering consumer *i* from all of the *k* warehouses

$f(W)$ – reliability of the system, the objective function is defined as the minimum value of $R_i$ - $f(W) = \min\{R_i \mid i \in C\}$

**COMPETING INTERESTS**

Authors have declared that no competing interests exist.

References


[1] Z. Drezner, Z. Hamacher *Facility Location - Applications and Theory*, Springer-Verlag, Berlin-Heidelberg, 2002

[2] L.V. Snyder, M.S. Daskin, C-P. Teo. *The Stochastic Location Model with Risk Pooling*, European Journal of Operational Research 179(3), pp. 1221-1238, 2007

[3] M.S. Daskin *What You Should Know About Location Modeling*, Wiley Periodicals, Inc. Naval Research Logistics 55, pp. 283–294, DOI 10.1002/nav.20284, 2008

[4] H.S. Hwang. *A Stochastic Set-Covering Location Model for Both Items Ameliorating and Deteriorating*, Computers&Industrial Engineering 46 (2), Special Issue on Selected Papers from the 27th International Conference on Computers and Industrial Engineering, Part 1. pp. 313-319, 2004.

[5] J. F. Alegre, A. Álvarez, S. Casado, and J. A Pacheco. *Use of Memetic Algorithms to Solve a Stochastic Location Model: Health resources for diabetics in some provinces of Castilla-Leon*. XIII Jornadas de ASEPUMA, 2005

[6] K. Cattani, G. Ferrer, W. Gilland, *Simultaneous Production of Market-Specific and Global Products: A Two-Stage Stochastic Program with Additional Demand after Recourse*, Wiley Periodicals, Inc. Naval Research Logistics 50, pp. 438–461, 2003

[7] G. Chen, M. Daskin, Z.J. Shen, S. Uryasev. *A New Model For Stochastic Facility Location Modeling*, Research Report, 2005

[8] L.V. Snyder, M.S. Daskin. *Reliability Models for Facility Location: The Expected Failure Cost Case*, Transportation Science 39(3), pp. 400-416, 2005

[9] K. Cattani, G. Ferrer, W. Gilland, *Simultaneous Production of Market-Specific and Global Products: A Two-Stage Stochastic Program with Additional Demand after Recourse*, Wiley Periodicals, Inc. Naval Research Logistics 50, pp. 438–461, 2003

University of Pristina-Kosovska Mitrovica, Faculty of Sciences and Mathematics, 38220, Kosovska Mitrovica, Serbia.
*E-mail address*: natasa.kontrec@pr.ac.rs
Department of Operation Research, Faculty of Organizational Sciences, University of Belgrade, 11000, Belgrade
*E-mail address*: biljana.panic@fon.bg.ac.rs
University of Pristina-Kosovska Mitrovica, Faculty of Sciences and Mathematics, 38220, Kosovska Mitrovica, Serbia.
*E-mail address*: marina.tosic@pr.ac.rs
Department of Operation Research, Faculty of Organizational Sciences, University of Belgrade, 11000, Belgrade
*E-mail address:* mirko.vujosevi@fon.bg.ac.rs

Publisher
Union od Mathematicians of Macedonia – ARMAGANKA

Editor in chief
Prof. d-r Aleksa Malcheski

CIP - Каталогизација во публикација
Национална и универзитетска библиотека "Св. Климент Охридски", Скопје

51(082)

PROCEEDINGS of CODEMA 2020 = Зборник на трудови од CODEMA 2020. -
Skopje : Armaganka, 2021. - 375 стр. : илустр. ; 25 см

Текст на мак. и англ. јазик. - Фусноти кон текстот. - Библиографија кон трудовите

ISBN 978-608-4904-09-0

а) Математика -- Зборници

COBISS.MK-ID 53570309