CLASSIFICATIONS OF SYSTEMS OF LINEAR EQUATIONS BASED ON ITS GEOMETRICAL INTERPETATIONS

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Abstract. In this paper, a classification of systems of $m \in \{1,2,3\}$ linear equations is given. Also, an idea for generalization of the classification for arbitrary number of equations is presented. These classes are described trough the geometric interpretations of the equations. It is proven that the classes are subclasses of the three classes determined by the generalized Cramer's rule.

1 CRAMER'S AND GEOMETRIC CLASSES

System of m linear equations with n unknowns ($m \times n$) over the set of the real numbers is the system of equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \qquad ... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases},$$

where m and n are positive integers, and a_{ij} and b_i are real numbers for $1 \le i \le m$, $1 \le j \le n$.

Let A be the matrix and $\overline{A}$ be the augmented matrix of the system $m \times n$, where $m \le n$. Let $D_{i_1 i_2 ... i_m}$, where $i_1, i_2, ..., i_m$ is a permutation without repetition of a class m of the set $I_{n+1} = \{1,2,...,n+1\}$, be a minor of $\overline{A}$ whose columns are matching with $i_1, i_2, ..., i_m$ column of $\overline{A}$, respectively. Clearly, $D_{i_1 i_2 ... i_m} \ne 0$ if and only if $D_{j_1 j_2 ... j_m} \ne 0$, where $j_1, j_2, ..., j_m$ is a permutation of $\{i_1, i_2, ..., i_m\}$ such that $j_1 < j_2 < ... < j_m$.

The generalized Cramer's rule [1], divides the systems $m \times n$, $m \le n$, into the following three classes:

1. There exists $D_{i_1 i_2 ... i_m} \ne 0$ for some $i_1, i_2, ..., i_m \in I_n$. Then the system has a unique solution for $m = n$, and infinite solutions for $m < n$, expressed through $n - m$ parameters.

Parameters can be taken to be x_k, $k \in I_n \setminus \{i_1, i_2, ..., i_m\}$ and the solutions can be expressed explicitly by using the Cramer's formulas:

$$x_{i_1} = \frac{D_{x_{i_1}}}{D_{i_1 i_2 ... i_m}},\ x_{i_2} = \frac{D_{x_{i_2}}}{D_{i_1 i_2 ... i_m}},\ ...,\ x_{i_n} = \frac{D_{x_{i_n}}}{D_{i_1 i_2 ... i_m}}.$$

2. $D_{i_1 i_2 \ldots i_m} = 0$ for every $i_1, i_2, \ldots, i_m \in I_n$ and $D_{i_1 i_2 \ldots i_{m-1}(n+1)} \neq 0$ for some $i_1, i_2, \ldots, i_{m-1} \in I_n$. In this case the system does not have solution.

3. $D_{i_1 i_2 \ldots i_m} = 0$ for every $i_1, i_2, \ldots, i_m \in I_{n+1}$. Then the system does not have solution or has infinitely many solutions, expressed through $n - r(A)$ parameters, where $r(A)$ is the rank of A and $n - r(A) \in \{n - m + 1, \ldots, n-1, n\}$.

These classes will be called **Cramer classes**.

If $m > n$ the system is equivalent with the system in which the coefficients before the unknowns of $n+1, n+2, \ldots, m$-th column are zeros. In that system the first class is empty, and for $m - 1 > n$ the second class is again empty. In that cases there exist two or one Cramer (nonempty) class.

Linear combinations of m linear equations with n unknowns,

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1,\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2,\ \ldots,$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m,$$

with respect to the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$, is the linear equation:

$$\lambda_1 \left(a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n - b_1\right) + \lambda_2 \left(a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n - b_2\right) + \ldots$$
$$+ \lambda_n \left(a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n - b_m\right) = 0.$$

Definition 1.1. Zero linear equation is the equation in which all coefficients are zero. **Contradictory** linear equation is the equation in which all coefficients of the system (before unknowns) are zero, while the constant term is different from zero. Zero and contradictory linear equations will be called **singular**. **Regular** (nonsingular) linear equation is an equation in which at least one of the coefficients of the system is different than zero.

Thus, the zero linear equation is

$$0 \cdot x_1 + 0 \cdot x_2 + \ldots + 0 \cdot x_n = 0 \text{ i.e. } 0 = 0,$$

and contradictory linear equations are

$$0 \cdot x_1 + 0 \cdot x_2 + \ldots + 0 \cdot x_n = a,\ a \neq 0, \text{ i.e. } 0 = a,\ a \neq 0.$$

We will say that two equations are **contradictory to each other** (one is contradictory to the other) if they are regular and do not have a common solution. Two equations are equivalent if they have same solutions.

Definitions 1.2. One linear equation has rank 1 or **is in a general position** if it is a regular. m **linear equations are in a general position** if they consist a $m-1$ linear equations in a general position and the m-th equation is not equivalent or contradictory to any linear combination of the $m-1$ equations.

The system of m linear equations has a rank k if contains k linear equation in a general position, and do not contain $k+1$ linear equations in a general position. The system of m linear equations is **nonsingular** if it has a rank m.

Clearly $k \le \min\{m,n\}$, so for $m > n$ there are no m linear equations in a general position.

Theorem 1.3. Let be given m linear equations with n unknowns, $m \le n$. Then:

1. There exists $D_{i_1^0 i_2^0 \ldots i_m^0} \ne 0$, for some $i_1^0, i_2^0, \ldots, i_m^0 \in I_n$ if and only if the m-th equations are in a general position.

2. $D_{i_1 i_2 \ldots i_m} = 0$ for every $i_1, i_2, \ldots, i_m \in I_n$ and there exists $D_{i_1^0 i_2^0 \ldots i_{m-1}^0 (n+1)} \ne 0$ for some $i_1^0, i_2^0, \ldots, i_{m-1}^0 \in I_n$ if and only if there exist a $m-1$ equations in a general position, and the m-th equation is a contradictory to some linear combination of the other equations.

3. $D_{i_1 i_2 \ldots i_m} = 0$ for every $i_1, i_2, \ldots, i_m \in I_{n+1}$ if and only if there exist a $m-1$ equations in a general position, and the m-th equation is an equivalent to some linear combination of the other equations, or do not exist a $m-1$ equations in a general position.

Proof. One linear equation is a regular if and only if there exists $D_{i^0} \ne 0$, for some $i^0 \in I_n$. Let $m-1$ linear equations are in a general position if and only if $D_{i_1^0 i_2^0 \ldots i_{m-1}^0} \ne 0$, for some $i_1^0, i_2^0, \ldots, i_m^0 \in I_n$.

Let be given m linear equations with n unknowns and let the first $m-1$ linear equations are in a general positions. It follows that $D_{i_1^0 i_2^0 \ldots i_{m-1}^0} \ne 0$, for some $i_1^0, i_2^0, \ldots, i_m^0 \in I_n$.

Since $D_{i_1^0 i_2^0 \ldots i_{m-1}^0} \ne 0$, the system

$$\begin{cases} \alpha_1 a_{1 i_1^0} + \alpha_2 a_{2 i_1^0} + \ldots + \alpha_{m-1} a_{m-1, i_1^0} = a_{m i_1^0} \\ \alpha_1 a_{1 i_2^0} + \alpha_2 a_{2 i_2^0} + \ldots + \alpha_{m-1} a_{m-1, i_2^0} = a_{m i_2^0} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_1 a_{1 i_{m-1}^0} + \alpha_2 a_{2 i_{m-1}^0} + \ldots + \alpha_{m-1} a_{m-1, i_{m-1}^0} = a_{m i_m^0} \end{cases}$$

has a unique solution $\left(\alpha_1^0, \alpha_2^0, \ldots, \alpha_{m-1}^0\right)$.

If

$$a_{mk} = \alpha_1^0 a_{1k} + \alpha_2^0 a_{2k} + \ldots + \alpha_{m-1}^0 a_{m-1,k} + \beta_k^0, \; k \in I_{n+1} \setminus \left\{ i_1^0, i_2^0, \ldots, i_{m-1}^0 \right\},$$

then

$$D_{i_1^0 i_2^0 \dots i_{m-1}^0 k} = \begin{vmatrix} a_{1i_1^0} & a_{1i_2^0} & \dots & a_{1i_{m-1}^0} & a_{1k} \\ a_{2i_1^0} & a_{2i_2^0} & \dots & a_{2i_{m-1}^0} & a_{2k} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m-1,i_1^0} & a_{m-1,i_2^0} & \dots & a_{m-1,i_{m-1}^0} & a_{m-1,k} \\ \sum_{i=1}^{m-1} \alpha_i^0 a_{ii_1^0} & \sum_{i=1}^{m-1} \alpha_i^0 a_{ii_2^0} & \dots & \sum_{i=1}^{m-1} \alpha_i^0 a_{ii_{m-1}^0} & \sum_{i=1}^{m-1} \alpha_i^0 a_{ik} + \beta_k^0 \end{vmatrix}.$$

$$= \begin{vmatrix} a_{1i_1^0} & a_{1i_2^0} & \dots & a_{1i_{m-1}^0} & a_{1k} \\ a_{2i_1^0} & a_{2i_2^0} & \dots & a_{2i_{m-1}^0} & a_{2k} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m-1,i_1^0} & a_{m-1,i_2^0} & \dots & a_{m-1,i_{m-1}^0} & a_{m-1,k} \\ 0 & 0 & \dots & 0 & \beta_k^0 \end{vmatrix} = \beta_k^0 D_{i_1^0 i_2^0 \dots i_{m-1}^0}$$

It follows that

$D_{i_1^0 i_2^0 \dots i_{m-1}^0 k} = 0$ if and only if $\beta_k^0 = 0$, for every $k \in I_{n+1} \setminus \{i_1^0, i_2^0, \dots, i_{m-1}^0\}$.

From the last conclusion it follows that 1, 2 and the first statement of 3, are true. The second statement of 3 is also true, since we consider a cofactor expansion along the m-th row.■

According to the rank of the systems $m \times n$, $m \le n$, they form $m+1$ (nonempty) class, systems with rank $0,1,2,\dots,m$. The systems with rank m form one class matching with the first Cramer class (case 1), systems with rank $m-1$ are in the second and third class (case 2 and 3), and all other systems with rank less then $m-1$ are in the third class. For $m > n$ the systems form $n+1$ (nonempty) class, systems with rank $0,1,2,\dots,n$. If $m = n-1$ they belong to the second or the third Cramer class (the first class is empty), and if $m > n-1$ they belong only to the the third class. We will describe below a finer classification.

The systems with a rank 0, are divided into $m+1$ class, determined of the number of the zero equations l, $l \in \{0,1,\dots,m\}$. Then the number of the contradictory equations is $m-l$.

The systems with a rank 1, contain an equation with rank 1 (the other equations are regular and equivalent to it, contradictory to it, zero or contradictory), and are divided into classes determined by $k_1, k_2, \dots, k_s, l$ where $k_1, k_2, \dots, k_s$, $k_1 \ge k_2 \ge \dots \ge k_s$ are the numbers of the groups of maximally equivalent regular equations, and l is the number of zero equations (the number of contradictory equations is $m - k_1 - k_2 - \dots - k_s - l$).

Example 1.4. 1) The systems with 3 equations and rank 0, belong to 4 classes: $\underline{0}$, $\underline{1}$, $\underline{2}$ и $\underline{3}$.

The systems with 3 equations and rank 1, belong to

$$3+2\cdot 2+1\cdot 3=10 \text{ classes:}$$

$$1,1,1;\ 2,1;\ 3;\ 1,1,\underline{0};\ 1,1,\underline{1};\ 2,\underline{0};\ 2,\underline{1};\ 1,\underline{0};\ 1,\underline{1} \text{ and } 1,\underline{2}$$

(we write $1,1,1$ for $1,1,1,\underline{0}$ i.e. when the number of zero equations is 0).

2) The systems with 7 equations and rank 0, belong to 8 classes, and from rank 1 on:

$$11+11\cdot 2+7\cdot 3+5\cdot 4+3\cdot 5+2\cdot 6+1\cdot 7=108$$

classes.

The systems with rank 2 contain two linear equations in a general position Σ_1 and Σ_2. The other equations are: equivalent or contradictory to some linear combination Λ of the two equations that has 0, 1 or 2 nonzero coefficient i.e. the other equations are: zero or contradictory; equivalent or contradictory to one of the two equations; and equivalent or contradictory to a equations of arbitrary equation of a linear combination different of the two equations. If the system contains at least 4 regular equations, the other equations can be equivalent or contradictory to a same or different equation of: the both equations and an arbitrary equation of the linear combinations different of the two equations (Σ, Σ_1 and Σ_2, Σ and Λ_1, Λ_1 and Λ_2 ($\Sigma \in \{\Sigma_1, \Sigma_2\}$, $\Lambda_1, \Lambda_2 \in lk(\Sigma_1, \Sigma_2) \setminus \{\Sigma_1, \Sigma_2\}$, $\Lambda_1 \neq \Lambda_2$, where $lk(\Sigma_1, \Sigma_2)$ is the set of all linear combinations of Σ_1 and Σ_2). By combining the cases, a finer classification can be defined.

Then we can classified the systems with rank 3 and etc.

According to the previous criteria, the systems with rank $m-1$ are divided into $2m$ classes:

-The systems that belong to the case 2 are divided into m classes, depending on m-th equation, which is contradictory to the linear combinations with $0,1,\ldots,m-1$ nonzero coefficients.

- The systems that belong to the first subcase of 3 are divided into m classes, depending on m-th equation, which is equivalent to the linear combinations with $0,1,\ldots,m-1$ nonzero coefficients.

From the above discussion, it follows that these classes contain all the systems. In next section, using symbols, we will describe the classes of the systems with one, two or three linear equations.

The classification is based on geometric interpretations on the equations, and therefore the classes are called **geometric classes**.

2 GEOMETRIC INTERPRETATIONS OF SYSTEMS WITH ONE, TWO AND THREE LINEAR EQUATIONS

2.1. Geometric interpretation of the linear equations. The contradictory linear equation is interpreted by an empty set. For $n=1$, the zero equation geometricaly is interpreted by a line, while a regular $ax=b$, $a \neq 0$, by a point in line. For $n=2$, the zero equation geometrically is interpreted by a plane, while a regular $ax+by=c$, where $a \neq 0$ or $b \neq 0$ by a line in a plane. For $n=3$, the zero equation is interpreted by a space, and a regular $ax+by+cz=d$, where $a \neq 0$, $b \neq 0$ or $c \neq 0$, by a plane in a space.

The symbols for an empty set, a point, a line, a plane and a space are, respectively:

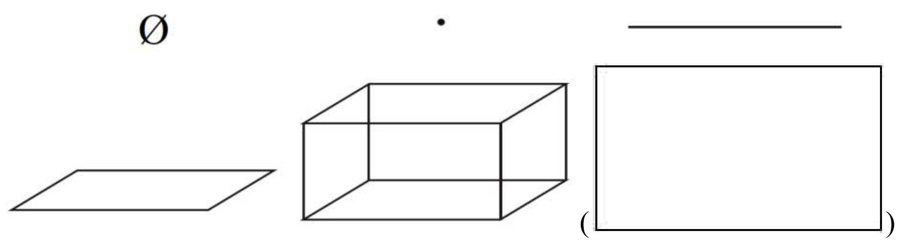
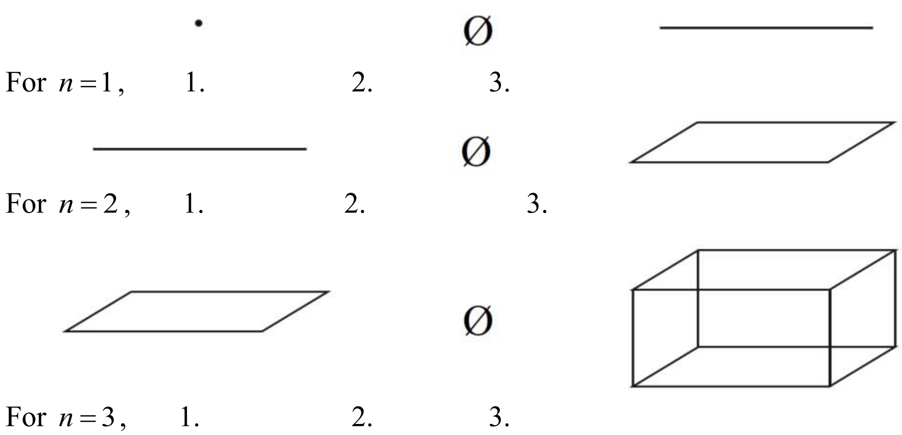


Figure 1

In Figure 2, geometric interpretations of the geometric classes are given.



1. there exists $D_{i_0} \neq 0$ for some $i_0 \in I_n$;

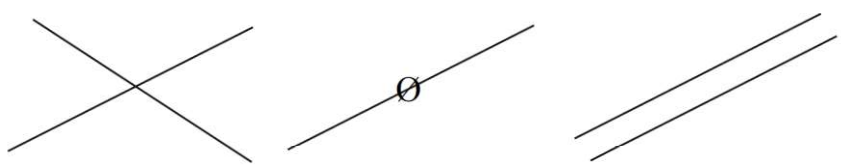
2. $D_i = 0$ for every $i \in I_n$ and $D_{n+1} \neq 0$; 3. $D_i = 0$ for every $i \in I_{n+1}$.

Figure 2

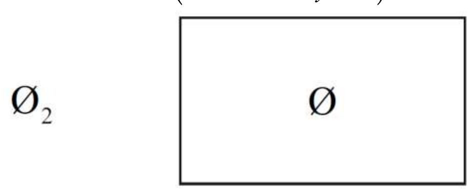
For $n>3$. The zero equation with n unknowns determines the n-dimensional space ($\mathbb{R}^n$), contradictory equation the empty set, while regular an $n-1$ dimensional plane. So, we can use the previous interpretation for $n=3$.

2.2. Geometric interpretation of systems of 2 linear equations. For the systems $m \times n$, $m,n \in \{2,3\}$, $m \le n$ the classification is given in the textbook [2]. In this paper we will generalize the classification for $m \in \{1,2,3\}$ and arbitrary $n \in \mathbb{N}$.

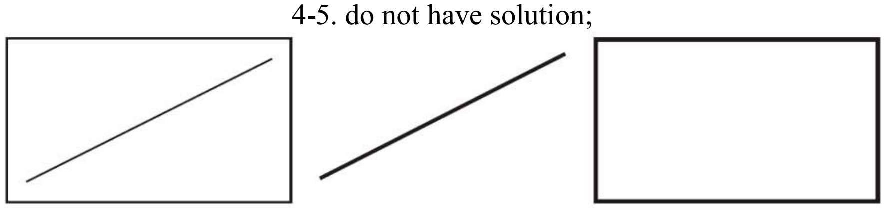
2×2. Two regular equations can be: in a general position, contradictory to each other, or equivalent. Their interpretations respectively are: two intersecting lines, two parallel lines or two matching lines. Therefore, there exist 8 class of the interpretations of the equations, given in Figure 3. Namely if the system has rank 0 then the two equations are zero (8), zero and contradictory (5), or contradictory (4). If the system has rank 1 and first of the equations is regular then the second equation is: equivalent to a linear combination of the first equation i.e. is zero (6) or regular and equivalent to the first equation (7); contradictory to linear combination of the first equation i.e. is contradictory (2) or regular and contradictory to the first equation (3). The last case is when the system has rank 2 or is nonsingular. In this case the equations are regular and have a unique common solution (1).



1. $D \neq 0$; 2-3. $D = 0 \wedge \left(D_x \neq 0 \vee D_y \neq 0\right)$;



4-5. do not have solution;

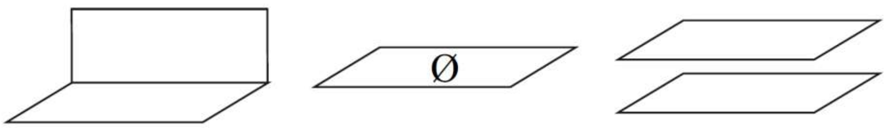


6-8. have infinite solutions expressed by: 6-7. one and 8. two parameters;

4-8. $D = D_x = D_y = 0$.

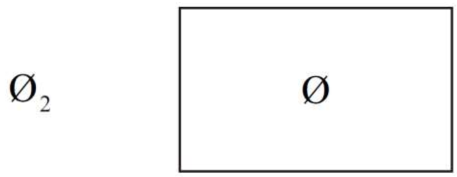
Figure 3

2×3. Two regular linear equations in general position, a contradictory to each other, or an equivalent are interpreted with two intersecting, parallel, or coincident planes, respectively. The classification is a restriction of the geometrical classification of the systems 2×3 that have 8 classes that are presented in Figure 4. For more visibility, the zero equation instead by a cuboid is presented by rectangular, a plane is presented by a parallelogram, and the lines are thickened in matching planes and spaces.

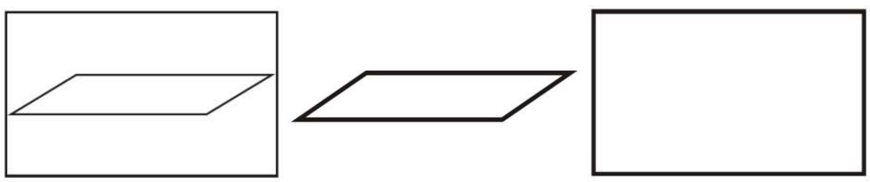


1. $D_{i_0 j_0} \neq 0$ for some $i_0, j_0 \in I_3$;

2-3. $D_{ij} = 0$ for every $i, j \in I_3$ and $D_{i_0 4} \neq 0$ for some $i_0 \in I_3$;



4-5. do not have solution;



6-8. have infinite solutions expressed by: 6-7 one and 8 three parameters;

4-8. $D_{ij} = 0$ for every $i, j \in I_4$, $i \neq j$.

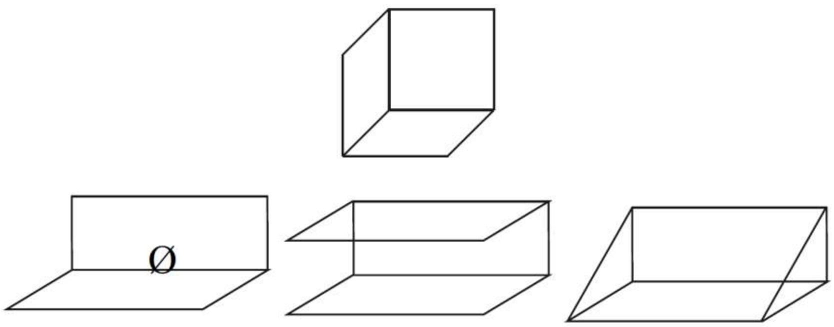
Figure 4

$2\times n$, $n>3$. Two regular linear equations that are in a general position, contradictory to each other, or equivalent are interpreted with two intersecting $n-1$-planes, parallel planes or coincident planes, respectively. There exist 8 mutual positions of the interpretations of the equations, which can be sketched by using one of the previous interpretations, for example for $n=3$.

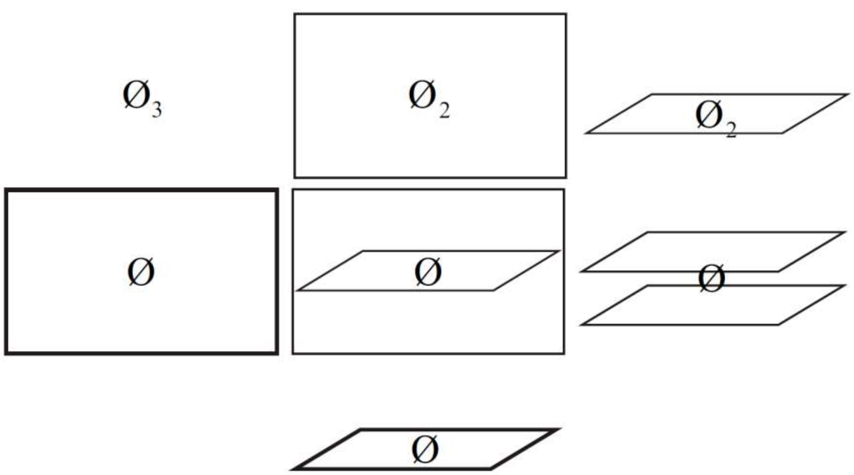
2×1. Two regular equations are contradictory to each other or equivalent are interpreted with two different points or two coinciding points. There exists 7 mutual positions of the interpretations of the equations (do not exist two equations in a general position, so the case 1 is not possible)

***2.3 Geometric interpretation of the system of 3 linear equations*.** 3×3 Three equations in a general position define three planes intersecting at a point (Figure 5, 1). Two equations in a general position define two intersecting, in a line, planes. Their linear combination, depending on the number of non-zero coefficients, determine: a space (15), a plane that coincides with one of the planes (17) or a plane different from the planes passing trough its common line (16). The contradictory equation to they linear combinations determines an empty set (2), a plane parallel to one of the planes (3) or a plane that intersects the planes in a lines parallel to its common line (4). Let the rank of the system be 1. If the three equations are regular, then the three equations are mutually equivalent (20), only one pair (14) none (13). If exactly two equations are regular, then they are equivalent (11,19) or contradictory to each other (10,12). If exactly one equation is regular, depending of the number of zero equations, 3 case are possible (7,9,18). Let the rank of the system is 0. Then depending of the number of zero equations, 4 cases are possible (5,6,8,21).

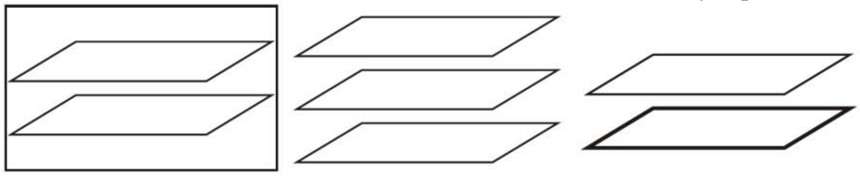
On Figure 5, the interpretations of all classes (a total 21), through symbols, sorted by Cramer's classes, and by the number of solutions of the systems are given. Of these, 8 classes are consisting of regular equations, determining 8 mutual positions of three planes in a space.



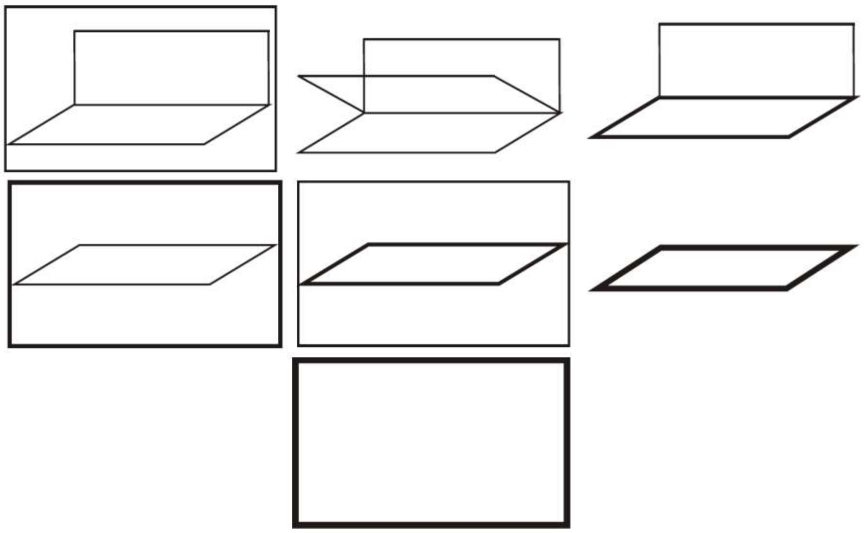
1. $D \neq 0$; 2-4. $D = 0 \wedge \left(D_x \neq 0 \vee D_y \neq 0 \vee D_z \neq 0 \right)$



5-11. do not have solution, such that have a contradictory equation



5-14. do not have solution



15-21. have infinite solutions expressed by: 15-17 one, 18-20 two, and 21 three parameters. 5-21. $D = D_x = D_y = D_z = 0$.

Figure 5

$\underline{3 \times n},\ n > 3$ For the systems $3 \times n$, $n > 3$ we can use interpretations for $n = 3$.

$\underline{3 \times 1}, \underline{3 \times 2}$. If $n = 2$ there exist 20, while for $n = 1$, 14 geometric interpretations of the geometric classes, which also can be described by words or can be sketched. Compared to $n = 3$, for $n = 2$ there are not three equations in a general position (falls out in the case 1), and for $n = 1$ does not exist even two equations in a general position (falls out in the cases 1-4 and 15-17).

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