A CONTRIBUTION TO THE LINEARIZATION OF THE VEKUA EQUATION

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Abstract. In the paper it is given a small contribution to the linearization of the Vekua differential equation.

1. INTRODUCTION

The equation

$$\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W} + F \qquad (1)$$

where $A = A(z)$, $B = B(z)$ and $F = F(z)$ are given complex functions from a complex variable $z \in D \subseteq \mathbb{C}$ is the well known Vekua equation [1] according to the unknown function $W = W(z) = u + iv$. The derivative on the left side of this equation has been introduced by G.V. Kolosov in 1909 [2]. During his work on a problem from the theory of elasticity, he introduced the expressions

$$\frac{1}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] = \frac{\hat{d}W}{dz} \qquad (2)$$

and

$$\frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] = \frac{\hat{d}W}{d\bar{z}} \qquad (3)$$

known as operator derivatives of a complex function $W = W(z) = u(x, y) + iv(x, y)$ from a complex variable $z = x + iy$ and $\bar{z} = x - iy$ corresponding. The operating rules for this derivatives are completely given in the monograph of Г. Н.Положий [3] (page18-31). In the mentioned monograph are defined so cold operator integrals $\hat{\int} f(z)dz$ and $\hat{\int} f(z)d\bar{z}$ from $z = x + iy$ and $\bar{z} = x - iy$ corresponding (page 32-41). As for the complex integration in the same monograph is emphasized that it is assumed that all operator integrals can be solved in the area D.

In the Vekua equation (1) the unknown function $W = W(z)$ is under the sign of a complex conjugation which is equivalent to the fact that $B = B(z)$ is

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not identically equaled to zero in D. That is why for (1) the quadratures that we have for the equations where the unknown function $W = W(z)$ is not under the sign of a complex conjugation, stop existing.

This equation is important not only for the fact that it came from a practical problem, but also because depending on the coefficients A, B and F the equation (1) defines different classes of generalized analytic functions. For example, for $F = F(z) \equiv 0$ in D the equation (1) i.e.

$$\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W} \qquad (4)$$

which is called canonical Vekua equation, defines so cold generalized analytic functions from fourth class; and for $A \equiv 0$ and $F \equiv 0$ in D, the equation (1) i.e. the equation $\frac{\hat{d}W}{d\bar{z}} = B\overline{W}$ defines so cold generalized analytic functions from third class or the (r+is)-analytic functions [3], [4].

Those are the cases when $B \neq 0$. But if we put $B \equiv 0$, we get the following special cases. In the case $A \equiv 0$, $B \equiv 0$ and $F \equiv 0$ in the working area $D \subseteq \mathbb{C}$ the equation (1) takes the following expression $\frac{\hat{d}W}{d\bar{z}} = 0$ and this equation, in the class of the functions $W = u(x, y) + iv(x, y)$ whose real and imaginary parts have unbroken partial derivatives u'_x, u'_y, v'_x and v'_y in D, is a complex writing of the Cauchy - Riemann conditions. In other words it defines the analytic functions in the sense of the classic theory of the analytic functions. In the case $B \equiv 0$ in D is the so cold areolar linear differential equation [3] (page 39-40) and it can be solved with quadratures.

2. MAIN RESULT

Let`s consider the Vekua equation

$$\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W} + F \qquad (1)$$

where $A = A(z), B = B(z)$ and $F = F(z)$ are given analytic functions from a complex variable $z \in D \subseteq \mathbb{C}$ and the areolar linear differential equation

$$\frac{\hat{d}W}{d\bar{z}} = AW + F \,. \qquad (5)$$

As mentioned above, the equation (5) can be solved and its solution is given with the following formula:

$$W = e^{\hat{\int} A(z) d\bar{z}} [\Phi(z) + \hat{\int} F(z) e^{-\hat{\int} A(z) d\bar{z}} d\bar{z}]. \qquad (6)$$

Here $\Phi = \Phi(z)$ is an arbitrary analytic function in the role of an integral constant.

If we have in mind that $A = A(z)$ and $F = F(z)$ are analytic functions, then they have the role of constants in the areolar integrals in (6) where the integration is by $\bar{z}$, we can write this solution in the following form

$$W = e^{A(z)\hat{\int} d\bar{z}} [\Phi(z) + F(z) \hat{\int} e^{-A(z)\hat{\int} d\bar{z}} d\bar{z}]$$

Now, $\hat{\int} d\bar{z} = \bar{z}$, so

$$W = e^{A\bar{z}} [\Phi(z) + F(z) \hat{\int} e^{-A\bar{z}} d\bar{z}]$$

and $\hat{\int} e^{-A\bar{z}} d\bar{z} = -\frac{1}{A} e^{-A\bar{z}}$, so we have

$$W = e^{A\bar{z}} [\Phi - \frac{F}{A} e^{-A\bar{z}}]$$

i.e.

$$W = \Phi e^{A\bar{z}} - \frac{F}{A}. \qquad (7)$$

The function $\Phi = \Phi(z)$ is an arbitrary analytic function in the role of an integral constant for the areolar linear differential equation (5). Let`s consider the possibility the function (7) to be a solution of the Vekua equation (1) if we put $\Phi = f(z) g(\bar{z})$, where $f = f(z)$ is an analytic function and $g = g(\bar{z})$ is an antianalytic function. In other words we would like to find the condition that will make the function

$$W = f(z) g(\bar{z}) e^{A\bar{z}} - \frac{F}{A} \qquad (8)$$

a solution of the Vekua equation (1).

For that purpose, we have to find the areolar derivative of this function and substitute it in the equation (1). We have

$$\frac{\hat{d}W}{d\bar{z}} = f(z) \frac{\hat{d}}{d\bar{z}} \left(g(\bar{z}) e^{A\bar{z}} \right) - 0$$

because A and F are analytic functions and

$$\frac{\hat{d}W}{d\bar{z}} = f(z)\left(\frac{\hat{d}g}{d\bar{z}}e^{A\bar{z}} + g(\bar{z})Ae^{A\bar{z}}\right) \text{ i.e.}$$

$$\frac{\hat{d}W}{d\bar{z}} = f(z)e^{A\bar{z}}\left(\frac{\hat{d}g}{d\bar{z}} + Ag(\bar{z})\right).$$

After substituting this derivative and the function (8) in (1), we get

$$f(z)e^{A\bar{z}}\left(\frac{\hat{d}g}{d\bar{z}} + Ag(\bar{z})\right) = A\left(f(z)g(\bar{z})e^{A\bar{z}} - \frac{F}{A}\right) + B\overline{\left(f(z)g(\bar{z})e^{A\bar{z}} - \frac{F}{A}\right)} + F$$

$$f(z)e^{A\bar{z}}\frac{\hat{d}g}{d\bar{z}} + Af(z)g(\bar{z})e^{A\bar{z}} = Af(z)g(\bar{z})e^{A\bar{z}} - F + B\overline{f(z)}\,\overline{g(\bar{z})}\,\overline{e^{A\bar{z}}} - B\frac{\bar{F}}{\bar{A}} + F$$

$$f(z)e^{A\bar{z}}\frac{\hat{d}g}{d\bar{z}} = B\overline{f(z)}\,\overline{g(\bar{z})}\,\overline{e^{A\bar{z}}} - B\frac{\bar{F}}{\bar{A}}. \qquad (9)$$

So, we can formulate the following

Theorem. *The function (8) is a solution of the Vekua equation (1) if the functions* $f = f(z)$ and $g = g(\bar{z})$ *are connected with the condition (9).*

What does this theorem gives us? We have the following method: if we have one Vekua equation (1), then, first we exclude the part with $\overline{W}$ and we solve the remaining linear areolar differential equation. Then, instead of the integral constant, we put a product of an analytic and an antianalytic function. If we consider this as a solution to the starting Vekua equation, then we can find the connection between the analytic and the antianalytic function. It is worth mentioning that if we put some concrete function or a class of functions in the place of $f = f(z)$**,** then for $g = g(\bar{z})$ we will get a new Vekua equation, where the unknown function is $g = g(\bar{z})$ which is not practical, and if we put some concrete function or a class of functions in the place of $g = g(\bar{z})$, then for $f = f(z)$ we can get an equation in which we have the functions f and $\bar{f}$.

Example. Let us consider the Vekua equation $\frac{\hat{d}W}{d\bar{z}} = zW + \overline{W}$. If we solve first the equation $\frac{\hat{d}W}{d\bar{z}} = zW$, we get the solution $W = \Phi e^{z\bar{z}}$. Now, if we proceed as described, we get $f(z)\frac{\hat{d}g}{d\bar{z}} = \overline{f(z)}\,\overline{g(\bar{z})}$. One solution are the functions $f = e^{z}$, $g = e^{\bar{z}}$.

Note: If $F=0$, i.e. instead the Vekua equation (1) we would like to consider the canonical Vekua equation (4), than the condition (9) would be a condition (10), where

$$f(z)e^{A\bar{z}}\frac{\hat{d}g}{d\bar{z}}=B\overline{f(z)}\overline{g(\bar{z})}\overline{e^{A\bar{z}}}\ . \qquad (10)$$

It is interesting that in this case, the modulo of B depends only from the antianalytic part of the integral constant, that is the function g and its areolar derivative, i.e.

$$|B|=\left|\frac{1}{g}\cdot\frac{\hat{d}g}{d\bar{z}}\right|.$$

In [6] it is considered the case of accordance of the equation (1) and the generalized linear differential equation. Here we consider the accordance with the areolar linear differential equation, which at the same time is a process of linearization of the Vekua equation. Since we take some restrictions, the result is just a contribution in it.

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