ONE THEOREM FOR ONE TYPE VEKUA EQUATION

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Abstract. In the paper one theorem for one type Vekua equation is proven.

1. INTRODUCTION

The equation

$$\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W} + F \qquad (1)$$

where $A = A(z)$, $B = B(z)$ and $F = F(z)$ are given complex functions from a complex variable $z \in D \subseteq \mathbb{C}$ is the well known Vekua equation [1] according to the unknown function $W = W(z) = u + iv$. The derivative on the left side of this equation has been introduced by G.V. Kolosov in 1909 [2]. During his work on a problem from the theory of elasticity, he introduced the expressions

$$\frac{1}{2}[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})] = \frac{\hat{d}W}{dz} \qquad (2)$$

and

$$\frac{1}{2}[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})] = \frac{\hat{d}W}{d\bar{z}} \qquad (3)$$

known as operator derivatives of a complex function $W = W(z) = u(x,y) + iv(x,y)$ from a complex variable $z = x + iy$ and $\bar{z} = x - iy$ corresponding. The operating rules for this derivatives are completely given in the monograph of Г. Н.Положий [3] (page18-31). In the mentioned monograph are defined so cold operator integrals $\hat{\int} f(z)dz$ and $\hat{\int} f(z)d\bar{z}$ from $z = x + iy$ and $\bar{z} = x - iy$ corresponding (page 32-41). As for the complex integration in the same monograph is emphasized that it is assumed that all operator integrals can be solved in the area D.

In the Vekua equation (1) the unknown function $W = W(z)$ is under the sign of a complex conjugation which is equivalent to the fact that $B = B(z)$ is not identically equaled to zero in D. That is why for (1) the quadratures that we have for the equations where the unknown function $W = W(z)$ is not under the sign of a complex conjugation, stop existing.

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This equation is important not only for the fact that it came from a practical problem, but also because depending on the coefficients A, B and F the equation (1) defines different classes of generalized analytic functions. For $F = F(z) \equiv 0$ in D the equation (1) defines so cold generalized analytic functions from fourth class; for $A \equiv 0$ and $F \equiv 0$ in D, the equation (1) defines so cold generalized analytic functions from third class or the (r+is)-analytic functions [3], [4]. Those are the cases when $B \neq 0$. But if we put $B \equiv 0$, we get the following special cases. In the case $A \equiv 0$, $B \equiv 0$ and $F \equiv 0$ in the working area $D \subseteq \mathbb{C}$ the equation (1) defines the analytic functions in the sense of the classic theory of the analytic functions. In the case $B \equiv 0$ in D is the so cold areolar linear differential equation [3] (page 39-40) and it can be solved with quadratures.

2. MAIN RESULT

Let`s consider the Vekua equation (1), where $A = 1$ and $B = 1$, i.e.

$$\frac{\hat{d}W}{d\bar{z}} = W + \overline{W} + F \tag{5}$$

where $F = F(z)$ is a given analytic function from a complex variable $z \in D \subseteq \mathbb{C}$. If we make a conjugation in (5), we get

$$\overline{\frac{\hat{d}W}{d\bar{z}}} = \overline{W} + W + \overline{F} \tag{6}$$

Now, lets add and subtract (5) and (6). We get

$$\frac{\hat{d}W}{d\bar{z}} + \overline{\frac{\hat{d}W}{d\bar{z}}} = 2\left(W + \overline{W}\right) + F + \overline{F} \tag{7}$$

$$\frac{\hat{d}W}{d\bar{z}} - \overline{\frac{\hat{d}W}{d\bar{z}}} = F - \overline{F} \tag{8}$$

If we have in mind the definition for $\frac{\hat{d}W}{d\bar{z}}$, (3), for $\overline{\frac{\hat{d}W}{d\bar{z}}}$ we have

$$\frac{\hat{d}W}{d\bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]$$

$$\overline{\frac{\hat{d}W}{d\bar{z}}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right].$$

So, for the left sides of (7) and (8) we get

$$\frac{\hat{d}W}{d\bar{z}} + \overline{\frac{\hat{d}W}{d\bar{z}}} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \tag{9}$$

$$\frac{\hat{d}W}{d\bar{z}} - \overline{\frac{\hat{d}W}{d\bar{z}}} = i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{10}$$

If we substitute (9) in (7) and (10) in (8) we get a system of equation

$$\begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 2\left(W + \overline{W}\right) + F + \overline{F} \\ i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = F - \overline{F} \end{cases}. \tag{11}$$

The unknown function is $W = W(z) = u + iv$, so $W + \overline{W} = 2u$. If $F = f_1 + if_2$, then $F + \overline{F} = 2f_1$ and $F - \overline{F} = 2if_2$. So, for the system (11) we have

$$\begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 4u + 2f_1 \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 2f_2 \end{cases}. \tag{12}$$

If we find the derivative by x in the first equation and the derivative by y in the second equation in (12), we get

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x \partial y} = 4\frac{\partial u}{\partial x} + 2\frac{\partial f_1}{\partial x} \\ \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2} = 2\frac{\partial f_2}{\partial y} \end{cases}$$

Now, we find the sum of the two equations in the last system and we get a partial differential equation from second order for $u = u(x, y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial u}{\partial x} + 2\frac{\partial f_1}{\partial x} + 2\frac{\partial f_2}{\partial y}. \tag{13}$$

If we apply the Furrier method, i.e. we suppose that the unknown function $u = u(x, y)$ can be written in the following form

$$u = P(x) \cdot G(y)$$

we get that $\frac{\partial u}{\partial x} = P'G, \quad \frac{\partial^2 u}{\partial x^2} = P''G$

and $\frac{\partial u}{\partial y} = PG', \quad \frac{\partial^2 u}{\partial y^2} = PG''$.

If we substitute this in (13), we get that

$$P''G + PG'' - 4P'G = 2\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right)$$

or

$$\left(P'' - 4P'\right)G + PG'' = 2\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right). \tag{14}$$

If $F = 0$, then (14) will have the following form

$$\left(P'' - 4P'\right)G = -PG''$$

and if we divide the variables

$$\frac{P''-4P'}{P}=-\frac{G''}{G}=\lambda^2=const.$$

we get ordinary linear differential equations from second order with constant coefficients:

$$P''-4P'-\lambda^2P=0 \qquad G''+\lambda^2G=0$$

$$r^2-4r-\lambda^2=0 \qquad r^2+\lambda^2=0$$

$$r_{1/2}=2\pm\sqrt{4+\lambda^2} \qquad r_{1/2}=\pm i\lambda$$

$$G(y)=C\cos\lambda y+D\sin\lambda y$$

$$P(x)=Ae^{(2+\sqrt{4+\lambda^2})x}+Be^{(2-\sqrt{4+\lambda^2})x}$$

So, for $u=u(x,y)$ we have

$$u(x,y)=\left(Ae^{(2+\sqrt{4+\lambda^2})x}+Be^{(2-\sqrt{4+\lambda^2})x}\right)\left(C\cos\lambda y+D\sin\lambda\right). \quad (15)$$

If we put this and $F=0$ in the second equation in (12), for the function $v=v(x,y)$ we have

$$\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=0$$

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=-\left(Ae^{(2+\sqrt{4+\lambda^2})x}+Be^{(2-\sqrt{4+\lambda^2})x}\right)\left(-C\lambda\sin\lambda y+D\lambda\cos\lambda\right)$$

$$v=\left(C\lambda\sin\lambda y-D\lambda\cos\lambda\right)\int\left(Ae^{(2+\sqrt{4+\lambda^2})x}+Be^{(2-\sqrt{4+\lambda^2})x}\right)dx+\varphi(y)$$

$$v=\left(C\lambda\sin\lambda y-D\lambda\cos\lambda\right)\left[\frac{A}{2+\sqrt{4+\lambda^2}}e^{(2+\sqrt{4+\lambda^2})x}+\right.$$
$$\left.+\frac{B}{2-\sqrt{4+\lambda^2}}e^{(2-\sqrt{4+\lambda^2})x}\right]+\varphi(y) \quad (16)$$

where $\varphi=\varphi(y)$ is an arbitrary function as an integral constant.

We have proven the following

Theorem. *The equation (5), where* $F=F(z)=f_1+if_2$ *is a given analytic function from a complex variable* $z\in D\subseteq\mathbb{C}$ *has a solution* $W=W(z)=u+iv$, *whose real part* $u=u(x,y)$ *satisfies the partial differential equation from second order (13). Moreover, if* $u=P(x)\cdot G(y)$, *then the functions* $P=P(x)$ *and* $G=G(y)$ *satisfy the equation (14). If* $F=0$

in (5), then the real and the imaginary part of the solution $W = W(z) = u + iv$ *of (5) are given with (15) and (16).*

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